Integrals

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In general, there are two type of integral problems: equalities and inequalities. In the two lectures, we will focus on equalities. Let us start with some standard techniques computing integrals.

Explore the symmetry

Example (Putnam 1987 B1). *Evaluate*

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}.$$

Example (Putnam and Beyond, Problem 453). *Compute the integral*

$$\int_{-1}^1 \frac{\sqrt[3]{x}}{\sqrt[3]{1-x}+\sqrt[3]{1+x}}dx.$$

Find the right substitution

Example (Putnam and Beyond, Problem 455). *Let a and b be positive real numbers. Compute*

$$\int_a^b \frac{e^{\frac{x}{a}}-e^{\frac{b}{x}}}{x}dx.$$

Trigonometric identities can be helpful

Example (Putnam and Beyond, Problem 458). *Compute the integral*

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx.$$

Hint: $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

Riemann sums

Example (Putnam and Beyond, Page 154). *Denote by G_n the geometric mean of the binomial coefficients*

$$\binom{n}{0},\binom{n}{1},\dots,\binom{n}{n}.$$

Prove that

$$\lim_{n\to\infty}\sqrt[n]{G_n}=\sqrt{e}.$$

Solution (Suggested by –). *We want to show that*

$$\lim_{n\to\infty} \frac{1}{n(n+1)} \sum_{i=1}^{n} \log \binom{n}{i} = \frac{1}{2}.$$

Use the Stolz–Cesàro theorem, the left-hand-side is equal to

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{n} \left(\log \binom{n+1}{i} - \log \binom{n}{i} \right) &= \lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{n} \left(\log \frac{n+1}{n+1-i} \right) \\
&= - \lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{n} \left(\log \frac{n+1-i}{n+1} \right) \\
&= - \int_0^1 \frac{1}{2} \log(1-x)dx \\
&= - \int_0^1 \frac{1}{2} \log x dx \\
&= -\frac{1}{2}(x \log x - x)|_0^1 \\
&= -\frac{1}{2}.
\end{aligned}$$

Here, using l'Hopital's rule, we have $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 0$, *which implies the last equality.*

Further exercises

1. (Putnam and Beyond, Page 150) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^{\pi} x f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx.$$

2. (Putnam and Beyond, Problem 457) Let a be a positive real number. Compute the integral

$$\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}.$$

3. (Putnam 1980, A3) Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

   Hint: this problem is indeed similar to the problem **??**.

4. (Putnam 1982, A3) Evaluate

$$\int_0^{\infty} \frac{\arctan(\pi x) - \arctan(x)}{x} dx.$$

5. (Putnam 1989, A2) Evaluate

$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dy dx,$$

   where a and b are positive.

6. (Putnam and Beyond, Problem 468) Compute

$$\lim_{n\to\infty}\left(\frac{1}{4n^2-1^2}+\frac{1}{4n^2-2^2}+\cdots+\frac{1}{4n^2-n^2}\right).$$

7. (Putnam and Beyond, Problem 447) Compute the indefinite integral

$$\int \frac{x^2+1}{x^4-x^2+1}dx.$$

8. (Putnam 1992, A2) Define $C(\alpha)$ to be the coefficient of x^1992 in the power series about $x=0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1\left(C(-y-1)\sum_{k=1}^{1992}\frac{1}{y+k}\right).$$

9. (Putnam 2016, A3) Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x)+f\left(1-\frac{1}{x}\right)=\arctan x$$

for all real number $x\neq 0$. Find

$$\int_0^1 f(x)dx.$$