Integrals

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In general, there are two type of integral problems: equalities and inequalities. In the two lectures, we will focus on equalities. Let us start with some standard techniques computing integrals.

Explore the symmetry

Example (Putnam 1987 B1)**.** *Evaluate*

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}.$$

Example (Putnam and Beyond, Problem 453)**.** *Compute the integral*

$$\int_{-1}^{1} \frac{\sqrt[3]{x}}{\sqrt[3]{1-x}+\sqrt[3]{1+x}}dx.$$

Find the right substitution

Example (Putnam and Beyond, Problem 455)**.** *Let a and b be positive real numbers. Compute*

$$\int_a^b \frac{e^{\frac{x}{a}}-e^{\frac{b}{x}}}{x}dx.$$

Trigonometric identities can be helpful

Example (Putnam and Beyond, Problem 458)**.** *Compute the integral*

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx.$$

Hint: $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

Riemann sums

Example (Putnam and Beyond, Page 154)**.** *Denote by G_n the geometric mean of the binomial coefficients*

$$\binom{n}{0},\binom{n}{1},\ldots,\binom{n}{n}.$$

Prove that

$$\lim_{n\to\infty}\sqrt[n]{G_n}=\sqrt{e}.$$

Further exercises

1. (Putnam and Beyond, Page 150) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that
$$\int_0^\pi xf(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx.$$

2. (Putnam and Beyond, Problem 457) Let a be a positive real number. Compute the integral
$$\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}.$$

3. (Putnam 1980, A3) Evaluate
$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$
Hint: this problem is indeed similar to the problem 2.

4. (Putnam 1982, A3) Evaluate
$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx.$$

5. (Putnam 1989, A2) Evaluate
$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dydx,$$
where a and b are positive.

6. (Putnam and Beyond, Problem 468) Compute
$$\lim_{n\to\infty} \left(\frac{1}{4n^2 - 1^2} + \frac{1}{4n^2 - 2^2} + \cdots + \frac{1}{4n^2 - n^2} \right).$$

7. (Putnam and Beyond, Problem 447) Compute the indefinite integral
$$\int \frac{x^2 + 1}{x^4 - x^2 + 1} dx.$$

8. (Putnam 1992, A2) Define $C(\alpha)$ to be the coefficient of x^1992 in the power series about $x = 0$ of $(1 + x)^\alpha$. Evaluate
$$\int_0^1 \left(C(-y - 1) \sum_{k=1}^{1992} \frac{1}{y + k} \right).$$

9. (Putnam 2016, A3) Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that
$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$
for all real number $x \neq 0$. Find
$$\int_0^1 f(x)dx.$$