For the last session, you solved this question from Summer 2012:

E0. Show that there exists an $\mathcal{N} \models \mathrm{PA}$ and an $a \in \mathcal{N} \smallsetminus \mathbb{N}$ so that a is definable in $\mathcal{N}$.

In a later qual, we asked a more sophisticated version:

E1. Let M be a model of PA that is not elementarily equivalent to $(\mathbb{N}, +, \cdot)$. Show that there is an infinite element of M that is definable.

What about nonstandard models of PA that *are* elementarily equivalent to $(\mathbb{N}, +, \cdot)$? In other words, what about nonstandard models of true arithmetic? This was never a qual problem, but for completeness:

E2. Let M be a model of True Arithmetic, i.e., the theory of $(\mathbb{N}, +, \cdot)$. Show that *no* infinite element of M is definable.

Incompleteness is intimately linked to computable inseparability.

E3. Consider the sets

$$A = \{\ulcorner\varphi\urcorner \colon \mathrm{PA} \vdash \varphi\},$$
$$B = \{\ulcorner\varphi\urcorner \colon \mathrm{PA} \vdash \neg\varphi\},$$

where $\ulcorner\varphi\urcorner$ is the Gödel code of the sentence φ. Show that A and B are computably inseparable. I.e., show that there is no computable set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

The next question extends the fact that no computably enumerable, consistent extension of PA is complete.

E4. Let T_0 and T_1 be computably enumerable, consistent extensions of PA (although, $T_0 \cup T_1$ need not be consistent). Show that there is a sentence ψ that is independent of both T_0 and T_1.

That's enough about PA and its extensions. The next few problems are basic model theory.

E5. Call a model M "nice" iff for every $a, b \in M$, there is an automorphism of M that moves a to b. Let T be a theory in a countable language. Show that if T has a nice model of some infinite cardinality, then T has nice models of all infinite cardinalities.

E6. Let L be a language which includes a unary relation symbol R. Let φ be an L-sentence and Γ a set of L-sentences neither of which contains the symbol R. If Γ proves φ in the language L, must there be a deduction of φ from Γ in which R does not occur (i.e., in the language $L - \{R\}$)? If so, prove that there is always such a deduction; and if not, describe Γ and φ which provide a counterexample.

E7. Let L be the language containing one binary relation symbol. A graph is a symmetric irreflexive binary relation. It is n-colorable iff there is a map from its universe into n such that no two elements in the relation are assigned the same value.

(a) Show that there is a first order L-theory T whose models are exactly the 3-colorable graphs.

(b) Prove that T is not finitely axiomatizable.

We finish with a strange combinatorial problem.

E8. Prove that there is no family $\{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}(\omega)$ such that for all $\alpha < \beta$: $A_\beta \setminus A_\alpha$ is infinite and $|A_\alpha \setminus A_\beta| \leq 7$.

E0 ans. By the Incompleteness Theorem, we can find a Δ_0-sentence $\varphi(x)$ such that $\mathbb{N} \models \forall x \, \neg\varphi(x)$ but $\mathrm{PA} + \exists x \, \varphi(x)$ is consistent. Then any model $\mathcal{N} \models \mathrm{PA} + \exists x \, \varphi(x)$ contains, by induction, a least witness a for φ, which must be both nonstandard and definable.

E1 ans. Let φ be a formula (in prenex normal form) of lowest quantifier-complexity so that $M \models \varphi$ and $\mathbb{N}$ does not. We observe that φ must begin with an $\exists$. In particular, φ cannot begin with a $\forall$. Otherwise, $\mathbb{N} \models \neg\varphi$, and $\neg\varphi = \exists x\psi$ where ψ is of lower quantifier-complexity. But then $\mathbb{N} \models \psi(x)$ for some x. Let $\hat{x} = 1+1+\cdots+1$ (i.e. the term which represents the element x). Then $\mathbb{N} \models \psi(\hat{x})$. But then this is a sentence of lower quantifier-complexity than φ, and thus $M \models \psi(\hat{x})$. Thus $M \models \varphi$. So, φ must be $\exists_n$ for some n. Let $\varphi = \exists x\psi$. Let $a \in M$ be the least witness for ψ. The induction axioms in PA give us that there is a least witness. This witness is definable. We need only conclude that it is infinite. Suppose towards a contradiction that x is finite. Then x is represented by a term $\hat{x} = 1 + 1 + \cdots + 1$. But then $M \models \psi(\hat{x})$. Since ψ is of lower quantifier-complexity than φ, we can conclude that $\mathbb{N} \models \psi(\hat{x})$, so $\mathbb{N} \models \varphi$, a contradiction.

E2 ans. If $\varphi(x)$ defines a element of M, then $(\exists! n) \, \varphi(n)$ holds in TA. Let n be the witness from $\mathbb{N}$. Then $\varphi(n)$ holds in M as well, so $\varphi(x)$ defines a finite element of M.

E3 ans. There are various ways to solve this problem. One is to use the fact (from class) that no complete consistent extension of PA is computable. Now show that from a separator C, we can compute a complete consistent extension of PA.

Another way: Use The Gödel Fixed Point Lemma. Suppose that C is a computable separator for A and B. Then let $\psi(x)$ be the formula that defines C. That is, for every x, $PA \vdash \psi(x)$ if and only if $x \in C$ and $PA \vdash \neg\psi(x)$ if and only if $x \notin C$. Then use the Gödel fixed point lemma to get a formula φ so that $PA \vdash \varphi \leftrightarrow \neg\psi(\ulcorner\varphi\urcorner)$. If $\ulcorner\varphi\urcorner \in C$, then $PA \vdash \psi(\ulcorner\varphi\urcorner)$, so $PA \vdash \neg\varphi$, so $\ulcorner\varphi\urcorner \in B$ contradicting that $B \cap C = \emptyset$. Similarly, if $\ulcorner\varphi\urcorner \notin C$, then $PA \vdash \neg\psi(\ulcorner\varphi\urcorner)$ and $PA \vdash \varphi$, so $\ulcorner\varphi\urcorner \in A \setminus C$, contradicting $A \subseteq C$.

E4 ans. Call disjoint c.e. sets A and B *computably inseparable* if there is no computable set C such that A is a subset of C and B is disjoint from C.

(Such a set C is called a *separator*.)

Given a c.e. set A, there is a Σ_1-formula φ_A in the language of arithmetic such that $n \in A$ if and only if $\mathrm{PA} \vdash \varphi_A(n)$. (Note that it's not true, in general, that $\mathrm{PA} \vdash \neg\varphi_A(n)$ for $n \notin A$. Indeed, this would mean that A is computable.)

Let A and B be computably inseparable c.e. sets. Let φ_A and φ_B be the corresponding formulas; more precisely, we modify $\varphi_A(n)$ to say that there is a witness that $n \in A$ which is $\leq$ the least witness that $n \in B$. Similarly, we modify $\varphi_B(n)$ to say that there is a witness that $n \in B$ which is $<$ the least witness that $n \in A$. Since A and B are disjoint, these modifications don't seem like they would do anything, but now we have:

$\mathrm{PA} \vdash \varphi_A(n) \to \neg\varphi_B(n)$ and, equivalently, $\mathrm{PA} \vdash \varphi_B(n) \to \neg\varphi_A(n)$.

Now let T_0 and T_1 be c.e. consistent extensions of PA. Such extensions can make new formulas of the form $\varphi_A(n)$ and $\varphi_B(n)$ true. Let A_0 be the set of n such that $T_0 \vdash \varphi_A(n)$. Define A_1, B_0, and B_1 similarly. Since $\mathrm{PA} \vdash \varphi_B(n) \to \neg\varphi_A(n)$, we know that for every $n \in B_0$, hence every $n \in B$, $T_0 \vdash \neg\varphi_A(n)$. The same holds for T_1.

Case 1: There is an n such that $\varphi_A(n)$ is independent of both T_0 and T_1. So we're done.

Case 2: No such n exists. We will get a contradiction in this case. Define a computable set C as follows. To decide if $n \in C$, enumerate all proofs from T_0 and T_1 until one of the theories is first seen to prove either $\varphi_A(n)$ or $\neg\varphi_A(n)$. This must happen eventually because we are in case 2. If we see a proof of $\varphi_A(n)$, we put $n \in C$. Otherwise, $n \notin C$.

Now note that C is a computable superset of A: If $n \in A$ then both theories prove $\varphi_A(n)$, hence can't prove $\neg\varphi_A(n)$. It's also disjoint from B: If $n \in B$ then both theories prove $\neg\varphi_A(n)$, hence neither can prove $\varphi_A(n)$. Therefore, C is a computable separator of A and B, which cannot exist.

E5 ans. Expand the language by adding a ternary function $A(x, y, z)$. The intent is that for each "fixed" x, y, $A(x, y, z)$ is an automorphism that moves x to y.

To formalize this, add an axiom saying that for each x, y, the map $z \mapsto A(x, y, z)$ is a permutation of the model moving x to y. Also, for each symbol of the language, add an axiom saying that this permutation is an automorphism with respect to that symbol. For example, if P is three-placed predicate, add an axiom saying that for all x, y, and all $z_1, z_2, z_3, w_1, w_2, w_3$: $w_1 =$

$A(x, y, z_1) \wedge w_2 = A(x, y, z_2) \wedge w_3 = A(x, y, z_3)$ implies that $P(z_1, z_2, z_3) \leftrightarrow P(w_1, w_2, w_3)$.

Then just apply the standard Löwenheim-Skolem Theorem to the new theory in the expanded language.

E6 ans. Straightforward application of the Completeness theorem: If Γ proves φ, then any model M of Γ is a model of φ. The same then also holds for any model M of Γ in the language $L - \{R\}$, so again by Completeness, there is a deduction of φ from Γ in the language $L - \{R\}$.

E7 ans. (a) For each $n \geq 3$ there is a first-order sentence which says that every subset of size n can be partitioned into three subsets none of which contains adjacent vertices. (b) For any odd $n > 1$ an n-cycle is not 2-colorable. Adding another point adjacent to all vertices in the n-cycle gives a graph which is not 3-colorable but every proper subgraph is.

E8 ans. Assume that we had such $\{A_\alpha \colon \alpha < \omega_1\}$. For each ξ, choose $B_\xi \subset A_{\xi+1} \backslash A_\xi$ with $|B_\xi| = 8$. Since $|[\omega]^8| = \aleph_0$, fix ξ, η such that $\xi < \xi + 1 < \eta < \eta + 1$ and $B_\xi = B_\eta$. Let $B = B_\xi = B_\eta$. Then $B \subseteq A_{\xi+1}$ and $B \cap A_\eta = \emptyset$, so $B \subseteq A_{\xi+1} \backslash A_\eta$, so $|A_{\xi+1} \backslash A_\eta| \geq 8$, which is a contradiction (taking $\alpha = \xi + 1$ and $\beta = \eta$).