Combinatorics

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Set theory

1. Given 2^{n-1} subsets of a set with n elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.

2. Let X be a subset of $\{1, 2, 3, \ldots, 2n\}$ with $n+1$ elements. Show that we can find $a, b \in X$ with a dividing b.

3. Let S be a finite set, and suppose that a collection F of subsets of S has the property that any two members of F have at least one element in common, but F cannot be extended (while keeping this property). Prove that F contains just half of the subsets of S.

4. Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

   (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;

   (b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ (for a, b, c distinct);

   (c) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

   Prove that there exists a one-to-one function g from A to R such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

5. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Geometric combinatorics

1. Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = \frac{1}{\sqrt{2}}$. Is this result true for a smaller k?

2. Show that if the points of the plane are colored black or white, then there exists an equilateral triangle whose vertices are colored by the same color.

3. Given a set M of $n \geq 3$ points in the plane such that any three points in M can be covered by a disk of radius 1, prove that the entire set M can be covered by a disk of radius 1.

4. Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

5. Is it possible to place infinitely many points in the plane in such a way that all pairwise distances have integer values and points are noncollinear?