Math 763. Homework 5
Due Thursday, October 31st

1. Prove that any finite set $X \subset \mathbb{A}^2$ is a complete intersection: the ideal $I(X) \subset k[\mathbb{A}^2]$ can be generated by two elements. (If all points have distinct x coordinates, the equations can be chosen in the form $y = f(x)$, $g(x) = 0$.) (Shafarevich, Problem I.6.4.)

2. Let $X \subset \mathbb{A}^3$ be the union of the coordinate axes; it is a (reducible) space curve. Prove that X is not a complete intersection: its ideal cannot be generated by two elements. (Shafarevich, Problem I.6.5.)

3. Suppose $f : X \to Y$ is a regular map between varieties. Suppose that all fibers of f are nonempty and have dimension d. Show that $\dim(X) = \dim(Y) + d$.

4. Let $Mat(n, m)$ be the vector space of $n \times m$ matrices. As an algebraic variety, it is isomorphic to $\mathbb{A}^{n+m}$. Fix r, and let

$$X \subset Mat(n, m)$$

be the subset of matrices of rank exactly r. Prove that X is an irreducible subvariety of $Mat(n, m)$ and find its dimension.

5. Prove that a generic degree d hypersurface in $\mathbb{A}^n$ contains no lines if $d > 3$ (and $n > 1$). More precisely, let V_d be the space of degree d polynomials in n variables. Prove that there exists a non-empty Zariski open subset $U \subset V_d$ such that for any $f \in U$, the hypersurface $f = 0$ contains no lines.

(This can be done by a more complicated kind of dimension-counting as follows. Let L be the variety of parametric lines in $\mathbb{A}^n$. Explicitly,

$$L = \mathbb{A}^n \times (\mathbb{A}^n - \{0\}),$$

where we view a pair $(v_0, v) \in L$ as a line parametrized by the vector-valued function $r(t) = v_0 + tv$. Consider the 'incidence subvariety' $Y \subset L \times V_d$ consisting of pairs $(v_0, v) \in L, f \in V_d$ such that f is identically zero on the line given by (v_0, v). Estimate the dimension of V_d by using two projections: $Y \to L$ and $Y \to V_d$.