Math 763. Homework 3

Due Thursday, October 3rd

In these problems (and everywhere else in the class), the ground field, which is denoted by k, is assumed to be algebraically closed.

Unless specified otherwise, we do not assume varieties to be either quasi-compact or separated (especially since the notion of separated-ness was not even discussed).

1. Denote by $G = GL(n, k)$ the group of all invertible $n \times n$ matrices with entries from k. Using the entries as coordinates, we embed $GL(n, k)$ into the affine space $\mathbb{A}^{n^2}$. It is easy to see that G is open.

Show that G is an *algebraic group*: the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ are regular. (Note that $G \times G$ is an open subset of $\mathbb{A}^{2n^2}$.)

2. Show that any quasi-compact variety is a noetherian topological space.

3. Let X be a variety. Show that X can be written as a locally finite union of irreducible components: there is a decomposition $X = \bigcup X_\alpha$ where each X_α is a closed irreducible subset, and each point of X has a neighborhood that meets only finitely many X_α's. The decomposition is unique if we assume that $X_\alpha \not\subset X_\beta$ for $\alpha \neq \beta$. (This follows from the previous problem if X is quasi-compact, so the problem is only interesting if X fails to be quasi-compact.)

4. Let X and Y be varieties and $f, g : X \to Y$ be regular maps. Prove that the subset

$$\{x \in X : f(x) = g(x)\} \subset X$$

is locally closed in X.

5. (The local ring of a point.) Let X be a variety; fix a point $x \in X$. Denote by O_x the *stalk* of the structure sheaf O_X at the point x. By definition, elements of O_x are equivalence classes of pairs (U, f), where U is a neighborhood of x and $f \in O_X(U)$ is a regular function; the pairs (U, f) and (U', f') are equivalent if x has a neighborhood $V \subset U \cap U'$ such that $f|_V = f'|_V$.

Show that O_x is a local ring (with respect to the natural operations on functions) and that if X is affine, O_x is the localization of $k[X]$ at the maximal ideal of x. (O_x is called the *local ring* of x; its elements are *germs* of regular functions.)

6. Show that the local ring is the universal local invariant of a point: the local rings of points $x \in X$ and $y \in Y$ are isomorphic as k-algebras if and only if there exist two neighborhoods $x \in U$ and $y \in V$ and an isomorphism $U \simeq V$ sending x to y.

7. Show that the local ring of a point $x \in X$ is integral if and only if x lies on a unique irreducible component of X.