Math 763. Homework 1
Due Thursday, September 19th

In these problems (and everywhere else in the class), the ground field, which is denoted by k, is assumed to be algebraically closed.

1. Show that the hyperbola $V(xy-1) \subset \mathbb{A}^2$ is not isomorphic to the affine line $\mathbb{A}^1$ (that is, that there is no bi-regular map between them; a bi-regular map is a regular bijection whose inverse is also regular).

2. Consider the cuspidal cubic $X = V(x^2 - y^3) \subset \mathbb{A}^2$. Prove that the map

$$\mathbb{A}^1 \to X : t \mapsto (t^2, t^3)$$

is bijective, but not bi-regular. (The map is also a homeomorphism in the Zariski topology.)

3. For two subsets $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, the cartesian product $X \times Y$ is naturally a subset of $\mathbb{A}^{n+m}$:

$$X \times Y = \{(a_1, \dots, a_{n+m}) | (a_1, \dots, a_n) \in X \text{ and } (a_{n+1}, \dots, a_{n+m}) \in Y\}.$$

Prove that if X and Y are algebraic, then so is $X \times Y$. Prove that

$$k[X \times Y] = k[X] \otimes k[Y].$$

4. Suppose that $f : X \to Y$ is a regular map between algebraic sets $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$. Prove that the graph

$$\Gamma_f = \{(P, f(P)) : P \in X\} \subset X \times Y \subset \mathbb{A}^{n+m}$$

is an algebraic set and that $\Gamma_f \simeq X$. (Shafarevich, Problem I.2.13)

5. A regular map of algebraic sets $f : X \to Y$ is said to be a *closed embedding* if $f(X)$ is an algebraic subset of Y and f induces an isomorphism between X and $f(X)$. Show that f is a closed embedding if and only if the induced map of algebras $f^* : k[Y] \to k[X]$ is surjective.

6. Set $X = \mathbb{A}^2$, and consider the regular map

$$\sigma : X \to X : (x, y) \mapsto (-x, -y).$$

Clearly, it is an involution: $\sigma^2 = id$. Define the quotient $Y = X/\sigma$ to be the affine variety whose coordinate ring $k[Y]$ is the algebra of invariants:

$$k[X]^\sigma = \{f \in k[X] : f \circ \sigma = f\}.$$

Describe Y explicitly by representing it as an algebraic set in an affine space. (Inspired by J. Ellenberg's colloquium talk a long time ago.)