35th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 26, 2013

Fill out the individual registration form

1. Let $I = 3\sqrt{2}\displaystyle\int_0^x \frac{\sqrt{1+\cos t}}{17-8\cos t}dt$. If $0 < x < \pi$ and $\tan I = \dfrac{2}{\sqrt{3}}$, what is x?

2. Let ABC be a right-angled triangle with $\angle ABC = 90^\circ$, and let D on AB such that $AD = 2DB$. What is the maximum possible value of $\angle ACD$?

3. Define a sequence (a_n) for $n \geq 1$ by $a_1 = 2$ and $a_{n+1} = a_n^{1+n^{-3/2}}$. Is (a_n) convergent (i.e. $\lim\limits_{n\to\infty} a_n < \infty$)?

4. A positive integer n is called *special* if it can be represented in the form $n = \dfrac{x^2+y^2}{u^2+v^2}$, for some positive integers x, y, u, v. Prove that

   (a) 25 is special;

   (b) 2013 is not special;

   (c) 2014 is not special.

5. Prove that $\dfrac{x}{\sqrt{1+x^2}} + \dfrac{y}{\sqrt{1+y^2}} + \dfrac{z}{\sqrt{1+z^2}} \leq \dfrac{3\sqrt{3}}{2}$ for any positive real numbers x, y, z such that $x+y+z = xyz$.

6. Let $X = \begin{pmatrix} 7 & 8 & 9 \\ 8 & -9 & -7 \\ -7 & -7 & 9 \end{pmatrix}$, $Y = \begin{pmatrix} 9 & 8 & -9 \\ 8 & -7 & 7 \\ 7 & 9 & 8 \end{pmatrix}$, let $A = Y^{-1} - X$ and let B be the inverse of $X^{-1} + A^{-1}$. Find a matrix M such that $M^2 = XY - BY$ (you may assume that A and $X^{-1} + A^{-1}$ are invertible).

7. Find $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2^n + 2^{-n})^2} + \frac{(-1)^n n}{(2^n - 2^{-n})^2}$.