SEVENTH ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. Let f be a bijection of a finite set X. Prove that the number of sets $A \subseteq X$ such that $f(A) = A$ is the integer power of 2.

Hint: Among all nonempty subsets A of X such that $f(A) = A$, the minimal ones form a partition of X.

2. Let

$$I(x) = \int_0^{\pi/2} \sin^x t dt, \quad \text{for} \quad x > 0.$$

Prove that for $x > 1$, the function $f(x) = xI(x)I(x-1)$ is periodic.

Hint: Use integration by parts to show $xI(x) = (x-1)I(x-2)$. Then prove the function $f(x)$ satisfies $f(x) = f(x-1)$.

3. Prove that there do not exists integers m, n, r such that $n(n+1)(n+2) = m^r$ and $n \geq 1, r \geq 2$.

Hint: Using the fact that $n+1$ is relatively prime to $n(n+2)$ to show that if $n(n+1)(n+2) = m^r$, then $n+1 = m_1^r$ and $n(n+2) = m_2^r$ for some positive integers m_1 and m_2. Then deduce that $m_2 < m_1^2 < m_2 + 1$, and hence a contradiction.

4. We are given a system of 10 linear equations in 50 variables $x_1, \ldots, x_{50}$:

$$a_{1,1}x_1 + a1,2x_2 + a_{1,3}x_3 + \cdots + a_{1,50}x_{50} = 0,$$

$$\ldots$$

$$a_{10,1}x_1 + a_{10,2}x_2 + a_{10,3}x_3 + \cdots + a_{10,50}x_{50} = 0.$$

Here all coefficients $a_{i,j}$ are two-digit numbers (in particular, positive integers). Show that there exists an integer solution $x_1, \ldots, x_{50}$ that is non-trivial (that is, not all x_i are zero) and such that $|x_i| < 10$.

Hint: First, plug in all possible $x_i = -4, -3, \ldots, 4$ to the 10 linear functions. Using pigeonhole principle to prove that there are two different sets of such $x_i's$ on which the 10 linear functions have the same values. Then the difference of the two sets of numbers gives a desired solution to the 10 equations.

5. Suppose that the continuous on $[0, 1]$ function f is positive at interior points and vanishes at the ends. Show that there exists a square whose two vertices lie on the x-axis, and the other two on the graph $y = f(x)$.

Solution: The problem is equivalent to find a number $a \in (0, 1)$ such that $f(a + f(a)) = f(a)$. Here we can assume $f(x) = 0$ for $x > 1$. Consider the function $F(x) = f(x+f(x)) - f(x)$. By extreme value theorem, there exists a number $c \in [0, 1]$ such that $f(x)$ achieves maximum value at c. By assumption, $c \in (0, 1)$. Therefore, $F(c) \leq 0$. Since $x + f(x)$ is continuous and has value 0 at 0, value 1 at 1, by intermediate value theorem, there exists

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$b \in (0,1)$ such that $b+f(b)=c$. Then $F(b) \leq 0$. Now, by intermediate theorem, $F(x)=0$ must have a solution between b and c, and hence a solution in $(0,1)$.

6. Let A be an $n \times n$ matrix, with entries chosen independently at random. The random law used to choose each entry of A may differ between entries. Show that if, for each column of A, the expected value of the sum of its entries is 0, then the expected value of $\det(A)$ is 0.

Solution: Let a_{ij} be the entry in row i column j of A. Let e_{ij} be the expected value of this entry and let $E=(e_{ij})$ be the matrix of expectations. The permutation expansion of $\det(A)$ gives that,

$$\det A = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) a_{1,\sigma(1)} \cdot \ldots \cdot a_{n,\sigma(n)}$$

Since the a_{ij} are chosen independently,

$$\mathbb{E}[\det A] = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \mathbb{E}[a_{1,\sigma(1)}] \cdot \ldots \cdot \mathbb{E}[a_{n,\sigma(n)}] = \det(E)$$

Since the expected value of the sum of the entries in each column is zero, linearity of expectation implies that the sum of entries in each column of E is 0. In other words, the sum of the rows in E, is zero and so the rows do not form a basis implying that $\det(E)=0$.