SIXTH ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION SOLUTIONS

1. Suppose $f(x)$, $g(x)$, $h(x)$ are differentiable functions on an interval $[a, b]$. Show that there exists $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$$

Solution. Let

$$\phi(x) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f(x) & g(x) & h(x) \end{vmatrix}.$$

Then $\phi(b) = \phi(a) = 0$. Therefore, there exists $c \in [a, b]$ such that $\phi'(c) = 0$. □

2. Suppose that N people sit at a round table. They get up and then sit back down again, not necessarily in their original chairs (but all chairs are occupied again). We would like to shift everybody at the table k sits clockwise so that at least two persons are in their original seats. Show that such k always exists if N is even, and conversely, if N is odd, such k may fail to exist. ($k = 0$ is allowed.)

Solution. Number the people by the initial position counter-clockwise by $1, 2, \dots, N$. When N is odd, if the new sitting positions are $N, N-1, \dots, 1$, then it is easy to check that after any rotation, exactly one person sits in the original seat.

For the remaining part, suppose that for some new sitting positions $a_1, \dots, a_N$ after any rotation there is exactly one person sitting in the original seat. We need to show that N is odd. Under the above assumption, modulo N, the sequence $a_1 - 1, a_2 - 2, \dots, a_N - N$ is a permutation of $0, 1, \dots, N-1$. Since $a_1, \dots, a_N$ is a permutation of $1, \dots, N$, we have

$$\sum_{i=1}^{N} a_i - i = 0.$$

Therefore, $0 + 1 + \cdots + (N-1)$ must be divisible by N, that is $N \mid \frac{(N-1)N}{2}$. This can only happen when N is odd.

□

3. Let $S = \{2^a 3^b \mid a, b \in \mathbb{Z}_{\geq 0}\}$. Find all solutions of $x + y = z$ with $x, y, z \in S$.

Solution. We call such a solution irreducible if $\gcd(x, y, z) = 1$. Notice that we can always multiply an irreducible solution (x, y, z) by a power of 2 and a power of 3 to obtain new solutions. Conversely, given a solution (x, y, z), the quotient $(\frac{x}{d}, \frac{y}{d}, \frac{z}{d})$ with $d = \gcd(x, y, z)$ is an irreducible solution. So it suffices to find all irreducible solutions.

Assume that (x, y, z) is an irreducible solution. Then two of x, y, z are odd. Moreover, one of x, y, z must be 1, otherwise 3 divides all x, y, z. Assume that $x = 1$, then one of y, z is not divisible by 3. So there are two possibilities: (i) $y = 2^m$ and $z = 3^n$, or (ii) $y = 3^n$ and $z = 2^m$.

For case (i), we have $1+2^m=3^n$. Except for $1+2=3$, all other solutions satisfy $m\geq 2$. For those solutions $8\mid 3^n-1$, which implies that n is even. Let $n=2n_0$. Then $2^m=(3^{n_0}-1)(3^{n_0}+1)$. The only two consecutive even numbers both being powers of 2 are 2 and 4. So the only remaining solution in this case is $1+8=9$.

For case (ii), we have $1+3^n=2^m$. Except for $1+1=2$, all other solutions satisfy $3\mid 2^m-1$, which implies that m is even. Let $m=2m_0$. Then $3^n=(2^{m_0}-1)(2^{m_0}+1)$. The only two consecutive odd numbers both being powers of 3 are 1 and 3. So the only remaining solution is $1+3=4$.

After all, all irreducible solutions are of the form $1+1=2$, $1+2=3$, $2+1=3$, $1+3=4$, $3+1=4$, $1+8=9$, $8+1=9$, and all the other solutions are obtained by multiplications of the above ones by a power of 2 and a power of 3. □

4. Three tennis players have a tournament according to the following scheme. In the first game, A plays with B, and from then on, the winner of the last game play with the participant who did not play in the last game. A person who wins two games in a row wins the tournament.

Assume that in each game, the two players have equal chance of winning. Find the probability of each participant's victory in the tournament.

Solution. Assume that at a moment of the game, X just win Y. Let $P(X),P(Y)$ and $P(Z)$ be the winning possibility of each player at this moment. In the next game, X will play with Z. If X wins the next game, then the tournament ends and X is the winner. If Z wins, then the game continues and the winning possibility becomes X: $P(Y)$, Y: $P(Z)$, and Z: $P(X)$. Therefore, we have the following relations

$$P(X)=\frac{1}{2}(1+P(Y)), P(Y)=\frac{1}{2}P(Z), \text{ and } P(Z)=\frac{1}{2}P(X).$$

Solving the equations, we have $P(X)=\frac{4}{7}$, $P(Y)=\frac{1}{7}$ and $P(Z)=\frac{2}{7}$.

No matter what is the outcome of the first game, C will be in the position of Z as in the above discussions. So C has 2/7 possibility winning the tournament. Clearly A and B have the same winning possibility, which has to be 5/14. □

5. A polynomial $f(x)$ of degree n satisfies

$$f(2^k)=k \qquad \text{for all } k=0,\dots,n.$$

Show that

$$f'(0)=2-\frac{1}{2^{n-1}}.$$

Solution. Let $g(x)=f(2x)-f(x)$. Then $g'(0)=f'(0)$. Moreover, $g(0)=0$ and $g(1)=g(2)=\cdots=g(2^{n-1})=1$. Since $g(x)-1$ is a polynomial of degree n and it has roots $1,2,\dots,2^{n-1}$, $g(x)-1=c(x-1)(x-2)\cdots(x-2^{n-1})$. Since $g(0)=0$,

$$c=\frac{-1}{(-1)(-2)\cdots(-2^{n+1})}, \quad \text{and hence} \quad g(x)=1-\frac{(x-1)(x-2)\cdots(x-2^{n-1})}{(-1)(-2)\cdots(-2^{n-1})}.$$

Therefore,

$$\begin{aligned} f'(0)=g'(0)&=-\frac{(-2)\cdots(-2^{n-1})}{(-1)(-2)\cdots(-2^{n-1})}-\frac{(-1)(-3)\cdots(-2^{n-1})}{(-1)(-2)\cdots(-2^{n-1})}-\cdots-\frac{(-1)(-2)\cdots(-2^{n-2})}{(-1)(-2)\cdots(-2^{n-1})}\\ &=1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}=2-\frac{1}{2^{n-1}}. \end{aligned}$$

□

6. Let $f : [0,1] \to [0,1]$ be a continuous bijective function such that $f(0) = 0$. Denote by f^{-1} the inverse function of f.

Show that for any $\alpha \geq 0$ the following inequality holds:

$$(\alpha + 2) \cdot \int_0^1 x^\alpha \left(f(x) + f^{-1}(x)\right)\, dx \leq 2.$$

Solution. Rewriting as double integrals, we have

$$\int_0^1 x^\alpha f(x)\, dx = \int_0^1 \int_0^{f(x)} x^\alpha dy dx,$$

and

$$\int_0^1 x^\alpha f^{-1}(x)\, dx = \int_0^1 \int_0^{f^{-1}(x)} x^\alpha dy dx = \int_0^1 \int_0^{f^{-1}(y)} y^\alpha dx dy.$$

Notice that the two regions of the double integrals $\int_0^1 \int_0^{f(x)} x^\alpha dy dx$ and $\int_0^1 \int_0^{f^{-1}(y)} y^\alpha dx dy$ exactly cover $[0,1] \times [0,1]$. Thus, we have

$$\int_0^1 \int_0^{f(x)} x^\alpha dy dx + \int_0^1 \int_0^{f^{-1}(y)} y^\alpha dx dy \leq \int_0^1 \int_0^1 \max\{x^\alpha, y^\alpha\} dx dy.$$

Clearly,

$$\int_0^1 \int_0^1 \max\{x^\alpha, y^\alpha\} dx dy = \int_0^1 \int_0^x x^\alpha dy dx + \int_0^1 \int_0^y y^\alpha dx dy = \frac{2}{\alpha + 2}.$$

The desired inequality follows from the above 4 displayed equations (and inequalities). □

7. Let n be a positive integer. As usual, let S_n the set of all permutations of the set $\{1, \ldots, n\}$. Prove that

$$\sum_{\sigma \in S_n} \frac{1}{\sigma(1)} \cdot \frac{1}{\sigma(1) + \sigma(2)} \cdots \frac{1}{\sigma(1) + \sigma(2) + \cdots + \sigma(n)} = \frac{1}{n!}.$$

Solution. We prove the following, more general statement. Let $x_1, \ldots, x_n$ be positive numbers. Then

$$\sum_{\sigma \in S_n} \frac{1}{x_{\sigma(1)}} \cdot \frac{1}{x_{\sigma(1)} + x_{\sigma(2)}} \cdots \frac{1}{x_{\sigma(1)} + x_{\sigma(2)} + \cdots + x_{\sigma(n)}} = \frac{1}{x_1 x_2 \cdots x_n}. \tag{1}$$

We prove this by induction on n. For $n = 1$ the statement is true. Assume that (**??**) holds for $n-1$, we will show it for n. Note that $x_{\sigma(1)} + x_{\sigma(2)} + \cdots + x_{\sigma(n)} = x_1 + \cdots + x_n$ and

$$\frac{x_1 + \cdots + x_n}{x_1 x_2 \cdots x_n} = \sum_{k=1}^n \frac{1}{x_1 x_2 \cdots \cancel{x_k} \cdots x_n}$$

Multiplying both sides of (**??**) with $x_1 + \cdots + x_n$, and using the same identity for $n - 1$ for the values of $x_1, \ldots, \cancel{x_k}, \ldots, x_n$ for each $1 \leq k \leq n$ we get the statement. □