TOPICS IN METRIC RIEMANNIAN GEOMETRY
(INCOMPLETE NOTES)

Contents

1. History and Overview

1.1. **History of comparison geometry and sphere theorems.** This course focuses on a central topic in Riemannian geometry which describes the relations between curvature and topology of the underlying space. We also emphasize advanced tools from metric geometry which regards a Riemannian manifold as a metric space and investigates global behaviors and geometric effects of geodesics in various senses.

$$\begin{tikzcd}
\text{Topology of manifold} \arrow[r, leftrightarrow, "\text{geodesics}"] \arrow[dr, leftrightarrow] & \text{Metric geometry} \arrow[d, leftrightarrow, "\text{Holonomy, } \nabla^2 r \text{ and } \Delta r, \text{ Jacobi equation, etc.}"] \\
 & \text{Curvature behaviors}
\end{tikzcd}$$

It is worth briefly mentioning the development of differential geometry in the history of mathematics. Differential geometry started with Gauß' famous work "*Disquisitiones Generales Circa Superficies Curvas*" (1827) which provides rigorous discussions of what we now call the *Gauß curvature* of a surface. The Gauß-Bonnet Theorem is probably the deepest theorem in differential geometry of surfaces.

Theorem 1.1 (Gauß-Bonnet)**.** *There are two versions.*

(1) *(Local version) Let T be a geodesic triangle in a surface Σ with three interior angles φ_1, φ_2, and φ_3. Then*
$$\sum_{j=1}^{3} \varphi_i - \pi = \int_T K d\sigma. \tag{1.1}$$

(2) *(Global version) Let Σ be a closed surface. Then*
$$\int_\Sigma K d\sigma = 2\pi\chi(\Sigma). \tag{1.2}$$

The rigorous form of Gauß-Bonnet theorem was first found in Wilhelm Blaschke's famous book "Vorlesungen über Differentialgeometrie" (1921).

The next essential step in the development of differential geometry is Riemannian's famous Habilitation "Über die Hypothesen, welche der Geometrie zu Grunde liegen" (10^{th} June, 1854) which represents the birth of *Riemannian geometry.* In this lecture, the intrinsic notions of what we call *Riemann curvature tensor* and *sectional curvature* nowadays were rigorously defined.

Definition 1.1 (Curvatures)**.** *Let (M^n, g) be a Riemannian manifold and let ∇ be its Levi-Civita connection. Then the Riemann curvature is defined to be*
$$\mathrm{Rm}(X,Y)Z \equiv \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z, \quad X, Y, Z \in \mathfrak{X}(M^n). \tag{1.3}$$

Let $X, Y \in T_pM^n$ two linearly independent tangent vectors. Then we define

$$\sec_g(X,Y) \equiv \frac{\langle \mathrm{Rm}(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2} \tag{1.4}$$

as the sectional curvature for the plane $\Pi_p \equiv \mathrm{Span}(X,Y) \subset T_pM^n$.

Now for any $p \in M^n$, let us take an orthonormal basis $\{E_1, \ldots, E_n\}$ of T_pM^n. Then we define

$$\mathrm{Ric}(X,Y) \equiv \left\langle \sum_{i=1}^{n} \mathrm{Rm}(E_i, X)Y, E_i \right\rangle \tag{1.5}$$

as the Ricci tensor at p.

In this course, we are interested in the manifolds with sectional or Ricci curvature uniformly bounded from below. In the comparison geometry of sectional curvature, the most powerful tool is Toponogov's triangle comparison theorem which is fundamentally important in the development of the field.

Theorem 1.2 (Toponogov)**.** *Let (M^n, g) be complete and $\sec_g \geq \kappa$. Let $p, q_-, q_+ \in M^n$ be three distinct points such that*

$$d_g(q_-, q_+) \leq \min\left\{d_g(p, q_-) + d_g(p, q_+), \pi/\sqrt{\kappa}\right\}. \tag{1.6}$$

Then there exists three distinct points $\bar{p}, \bar{q}_-, \bar{q}_+ \in (M^2_\kappa, \bar{d})$ such that

$$\bar{d}(\bar{p}, \bar{q}_-) = d_g(p, q_-),\ \ \bar{d}(\bar{p}, \bar{q}_+) = d_g(p, q_+),\ \ \bar{d}(\bar{q}_-, \bar{q}_+) = d_g(q_-, q_+), \tag{1.7}$$

and

$$d_g(p, \gamma_{q_-q_+}(t)) \geq \bar{d}(\bar{p}, \bar{\gamma}_{\bar{q}_-\bar{q}_+}(t)), \quad \forall t \in [0, d_g(q_-, q_+)], \tag{1.8}$$

where $\gamma_{q_-q_+}$ is a minimizing connecting q_- and q_+, and $\bar{\gamma}_{\bar{q}_-\bar{q}_+}$ is a minimizing geodesic connecting $\bar{q}_-$ and $\bar{q}_+$.

The study of such spaces started from formulations of *Sphere Theorems* with marvelous applications of comparison theorems due to Rauch in 1950s. Sphere is the simplest closed manifold. A fundamental problem is to ask how to use curvature condition to characterize a sphere, i.e., proving topological or geometric rigidity of sphere.

In general, under certain curvature condition $\mathscr{K}$, a *rigidity theorem* associated with a geometric quantity $\mathcal{Q}(M^n)$ looks like

$$\mathscr{K} \Longrightarrow \mathcal{Q}(M^n) \leq Q_0 \quad \text{with} \quad \mathcal{Q}(M^n) = Q_0 \Longleftrightarrow M^n \equiv \mathscr{M}, \tag{1.9}$$

where $\mathscr{M}$ is some model space. A further question in global Riemannian geometry is whether the quantity $\mathcal{Q}$ is promising to classify all manifolds that satisfy the curvature condition $\mathscr{K}$. In other words, what can we say when the quantity $\mathcal{Q}(M^n)$ fails to be maximal. More concretely, one wants to find out how continuous change of those "bounded" geometric affects the geometric shape or global topology of the underlying space, such as stability problem, almost rigidity problem. The formulation of *geometric/topological stability* is given as follows: under the curvature condition $\mathscr{K}$,

$$\mathcal{Q}(M^n) \geq Q_0 - \epsilon \Longrightarrow M^n \approx \mathscr{M}. \tag{1.10}$$

A similar problem, called *quantitative rigidity*, reveals more geometric information: find a suitable "distance"

$$\mathrm{Dist} : \mathfrak{M}(\mathscr{K}) \times \mathfrak{M}(\mathscr{K}) \to [0, \infty) \tag{1.11}$$

between manifolds that satisfy $\mathscr{K}$ such taht

$$\mathcal{Q}(M^n) \geq Q_0 - \epsilon \Longrightarrow \mathrm{Dist}(M^n, \mathscr{M}) \leq \tau(\epsilon), \quad \lim_{\epsilon \to 0} \tau(\epsilon) = 0. \tag{1.12}$$

Further, one is interested in whether $\mathrm{Dist}(M^n, \mathscr{M}) < \epsilon$ implies certain topological closeness.

One of Rauch's contributions in differential geometry is the following sphere theorem.

Theorem 1.3 (Rauch 1951)**.** *Given $n \geq 2$, there exists a dimensional constant $\epsilon = \epsilon_n > 0$ such that if a Riemannian manifold (M^n, g) satisfies $\pi_1(M^n) = \{e\}$ and $1 - \epsilon \leq \sec_g \leq 1$, then M^n is homeomorphic to a sphere.*

Another sphere theorem that inspired the developments of global Riemannian geometry is the *Quarter Pinching Sphere Theorem*.

Theorem 1.4 (Klingenberg and Berger, 1960)**.** *If a simply connected Riemannian manifold (M^n, g) satisfies $1/4 < \sec_g \leq 1$, then M^n is homeomorphic to a sphere. If M^n satisfies $1/4 \leq \sec_g \leq 1$, then M^n is either homeomorphic to a sphere or isometric to a symmetric space of compact type.*

Remark 1.1. *Brendle-Schoen and Lei Ni upgraded homeomorphism to diffeomorphism.*

Theorem 1.5. *Let (M^n, g) be complete and $\mathrm{Ric}_g \geq n - 1$. Then*

1. *(Bonnet-Myers) $\mathrm{diam}_g(M^n) \leq \pi$; (S.Y. Cheng 1975) equality holds iff M^n is isometric to $\mathbb{S}^n$.*
2. *(Bishop) $\mathrm{Vol}_g(M^n) \leq \mathrm{Vol}_g(\mathbb{S}^n)$ and equality holds iff M^n is isometric to $\mathbb{S}^n$.*

The following *Diameter Sphere Theorem* proves topological rigidity of manifolds with positive sectional curvature and large diameter.

Theorem 1.6 (Grove-Shiohama 1977)**.** *Let (M^n, g) satisfy $\sec_g \geq 1$ and $\mathrm{diam}_g(M^n) > \frac{\pi}{2}$. Then M^n is homeomorphic to a sphere.*

Remark 1.2. *M. Anderson (1990) constructed metrics on $\mathbb{C}P^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ with positive Ricci curvature and almost maximal diameter. In the case of $\sec_g \geq 1$, the lower bound $\frac{\pi}{2}$ is optimal since $\mathbb{R}P^n$ satisfies $\sec_g = 1$ and $\mathrm{diam}_g(\mathbb{R}P^n) = \frac{\pi}{2}$, and the Fubini-Study metric one $\mathbb{C}P^2$ satisfies $\sec_{g_{FS}} \geq 1$ and $\mathrm{diam}_{g_{FS}}(\mathbb{C}P^2) = \frac{\pi}{2}$.*

1.2. **Spaces of Riemannian manifolds.** In 1970, Cheeger proved a *Diffeomorphism Finiteness Theorem* which gives another flavor of statements other than sphere theorems and provides entirely new geometric point of view to be further established.

Theorem 1.7 (Cheeger 1970)**.** *Given $n \geq 2$, $\Lambda > 0$, $D > 0$ and $v > 0$, there exists a uniform constant $N = N(n, \Lambda, D, v) > 0$ such that the class*

$$\mathcal{M}(n, \Lambda, D, v) \equiv \{(M^n, g) : |\sec_g| \leq \Lambda,\ \mathrm{diam}_g(M^n) \leq D,\ \mathrm{Vol}_g(M^n) \geq v\} \tag{1.13}$$

has finitely many diffeomorphism types and this number is at most N.

Theorem 1.8 (Stability)**.** *Given a closed manifold (N^n, h), there exists $\delta = \delta(n, N^n) > 0$ such that if (M^n, g) satisfies*

$$\mathrm{dist}_{C^1}(M^n, N^n) < \delta, \tag{1.14}$$

then M^n is diffeomorphic to N^n.

Later in 1970s, Gromov created numerous new concepts and tools with further developments of comparison geometry, by which he proved many profound theorems such as *Almost Flat Manifold Theorem* and *Betti Numbers Estimates Theorem.* Cheeger and Gromov's theorems became the turning point of the development of Riemannian geometry. Based on such tremendous results, global Riemannian geometry become a very active and rapidly growing area in 1980s. Especially Gromov's Hausdorff convergence theory and Cheeger-Gromov's theory of collapsing Riemannian manifolds changed the shape of Riemannian geometry.

1.3. **Non-collapsing geometry of the Ricci curvature.** Since 1990s, emphasis of Riemannian geometry moved to studying the geometry of Ricci curvature, including the geometry and moduli space of Einstein metrics, Ricci flow, Hausdorff convergence under Ricci curvature bounds, etc.

Let us denote

$$\begin{aligned} \mathcal{M}^+_{\mathrm{Ric}}(n, \kappa) &\equiv \{\text{isometry class of } (M^n, g) : \mathrm{Ric}_g \geq \kappa\}\,, \\ \mathcal{M}et &\equiv \{\text{all metric spaces}\}\,. \end{aligned} \tag{1.15}$$

Theorem 1.9 (Gromov's Precompactness Theorem 1980)**.** *$\mathcal{M}^+_{\mathrm{Ric}}(n, \kappa)$ is precompact in $(\mathcal{M}et, d_{GH})$.*

By Gromov's Precompactness Theorem, for any sequence with Ricci curvature uniformly bounded below, one can always find a subsequence that converges to a metric space. For example, Gromov-Hausdorff limits of Einstein manifolds can be regarded as weak solutions of the Einstein equation. These geometric objects play the roles that distribution theory plays in analysis.

A fundamental problem asks how far the limit metric space differs from a smooth manifold, and what kind of singularities may appear.

Theorem 1.10. *Let $(M_j^n, g_j) \xrightarrow{GH} (X_\infty^n, d_\infty)$ be a non-collapsing sequence.*

(1) *(Anderson, Greene-Wu) If $|\operatorname{Rm}_{g_j}| \leq 1$, then X_∞^n is a smooth manifold and passing to a subsequence $(M_j^n, g_j) \xrightarrow{C^{1,\alpha}} (X_\infty^n, d_\infty)$ for any $\alpha \in (0,1)$.*
(2) *(Bando-Kasue-Nakajima) If $|\operatorname{Ric}_{g_j}| \leq n-1$ and $\int_{M_j^4} |\operatorname{Rm}_{g_j}|^{\frac{n}{2}} \leq \Lambda$, then X_∞^n is smooth away from finite orbifold singularities of number $\leq Q = Q(\Lambda, n)$.*

Bando-Kasue-Nakajima's regularity result for bounded Ricci curvature in the case of $n = 4$ is particularly interesting since the integral curvature bound is related the of Euler characteristic of the underlying space. Indeed, Chern-Gauß-Bonnet Theorem in dimension 4 can be written as

$$\chi(M^4) = \int_{M^4} P_\chi, \tag{1.16}$$

where

$$P_\chi = \frac{1}{8\pi^2}(|\operatorname{Rm}_g|^2 - 4|\operatorname{Ric}_g|^2 + |\operatorname{Scal}_g|^2). \tag{1.17}$$

In the Einstein case, one has

$$\chi(M^4) = \frac{1}{8\pi^2}\int_{M^4} |\operatorname{Rm}_g|^2, \tag{1.18}$$

It is worth mentioning that the above regularity theorem plays a fundamental role in the existence problem of Kähler-Einstein metrics on Fano surfaces. Since then, more investigations on the limit structure of spaces with Ricci curvature bounds appeared in the literature, which became a central and difficult topic in Riemannian geometry.

In the case of non-collapsing Riemannian manifolds, tremendous progress was made due to Cheeger-Colding's series of works.

Theorem 1.11 (Cheeger-Colding 1996)**.** *Given a smooth manifold (N^n, h) and any number $\kappa \in \mathbb{R}$, there exists $\delta = \delta(n, N^n, \kappa) > 0$ such that if (M^n, g) satisfies $\operatorname{Ric}_g \geq \kappa g$ and $d_{GH}(M^n, N^n) < \delta$, then M^n is diffeomorphic to N^n.*

Theorem 1.12 (Cheeger-Colding 1996)**.** *For any $n \geq 2$ and $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that if (M^n, g) satisfies $\operatorname{Ric}_g \geq n-1$ and $\operatorname{Vol}_g(M^n) \geq \operatorname{Vol}(\mathbb{S}^n) - \delta$, then M^n is diffeomorphic to $\mathbb{S}^n$.*

Theorem 1.13 (Cheeger-Colding 1997)**.** *Let (M_j^n, g_j) be a sequence of Riemannian manifolds such that $|\operatorname{Ric}_{g_j}| \leq n-1$ and $\operatorname{Vol}_{g_j}(B_1(x_j)) \geq v_0 > 0$. Then $(M_j, g_j) \xrightarrow{GH} (X_\infty^n, d_\infty)$ such that*

(1) *there exists a closed subset $\mathcal{S} \subset X_\infty^n$ with $\dim_{\mathcal{H}}(\mathcal{S}) \leq n-2$;*
(2) *$X_\infty^n \setminus \mathcal{S}$ is a smooth manifold with a $C^{1,\alpha}$-Riemannian metric.*

For non-collapsing manifolds with Ricci curvature uniformly bounded below, a crucial technical tool is *Cheeger-Colding's Metric Cone Structure Theorem.* It essentially states that

Theorem 1.14 (Cheeger-Colding 1996)**.** *For any $n \geq 2$, $v > 0$, and $\epsilon > 0$, there exists $Q = Q(n, v, \epsilon) > 0$ such that the following holds. If (M^n, g, p) satisfies $\mathrm{Ric}_g \geq \kappa$ and $\mathrm{Vol}_g(B_1(p)) \geq v > 0$, then for any $x \in B_2(p)$, there exists a metric space (Z, d_C, z) which is dilation invariant at z, i.e., a metric cone such that*

$$d_{GH}(B_r(p), B_r(z)) < \epsilon r \tag{1.19}$$

for all but finitely many bad scales $r_j \equiv 2^{-j}$ with $j \leq Q$.

Theorem 1.15 (Cheeger 2002)**.** *Let (M^n_j, g_j, p_j) be a sequence satisfying $\mathrm{Ric}_{g_j} \geq -(n-1)$, $\mathrm{Vol}_{g_j}(B_1(p_j)) \geq v > 0$, and*

$$\int_{B_2(p_j)} |\operatorname{Rm}_{g_j}|^p \leq \Lambda, \tag{1.20}$$

such that $(M^n_j, g_j) \xrightarrow{GH} (X^n_\infty, d_\infty)$. Then $\dim_{\mathcal{H}}(\mathcal{S}) \leq n - 2p$. In particular, in the Kähler case, $\dim_{\mathcal{H}}(\mathcal{S}) \leq n - 4$.

The next groundbreaking result is Cheeger-Naber's resolution of the *Codimension-4 Conjecture* for non-collapsing Einstein manifolds, which we will introduce in later lectures.

1.4. **Collapsing manifolds with bounded sectional curvature.** As we mentioned, the compactness and regularity results for non-collapsing Einstein manifolds are substantial techniques in studying the existence problem of Kähler-Einstein metrics. Compared with the rather complete theory established in the volume non-collapsing context, general theory is not available and very rare results have been known in the collapsed setting. However, such a theory is being demanded in many different subjects of geometry and physics. It is worth mentioning collapsing theory provided an important tool in completing the Perel'man's proof of *Thurston's Geometrization Conjecture*. Indeed, the last step of the proof is due to *Perel'man's Collapsing Theorem* for 3-manifolds.

Now we summarize substantial developments in the collapsing geometry of spaces with sectional curvature bounds. Let us consider a basic scenario in which a sequence of Riemannian manifolds (M^n_j, g_j) collapsing to a lower dimensional metric space $(M^n_j, g_j) \xrightarrow{GH} (X_\infty, d_\infty)$ with sectional curvature uniformly bounded $|\sec_{g_j}| \leq 1$. A primary task is to characterize the following extremal case. The *maximally collapsed* case happens when the limit space is a *single point*. This amounts to saying (M^n_j, g_j) satisfies $|\sec_{g_j}| \leq 1$ and $\mathrm{diam}_{g_j}(M^n_j) \to 0$. In a more scale-invariant version, we make the following notion. A closed Riemannian manifold is called ϵ-*almost-flat* if

$$\mathrm{diam}_{g_j}(M_j^n)^2 \cdot \max_{M_j^n} |\sec_{g_j}| \to 0. \tag{1.21}$$

Before discussing the structure of almost flat manifolds, we first introduce the Biberbach Theorem which characterizes the Riemannian geometry of flat manifolds.

Theorem 1.16 (Bieberbach, 1911)**.** *Any cocompact lattice $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n)$ must be a finite extension of $\mathbb{Z}^n$, i.e., Γ satisfies the following exact sequence*

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow H \to 1. \tag{1.22}$$

Here $H = \Gamma/\mathbb{Z}^n$ is called the holonomy group and satisfies $|H| \leq C(n)$. Equivalently, every closed flat manifold M^n is finitely covered by $\mathbb{T}^n$.

A groundbreaking result in collapsing theory is *Gromov's Almost Flat Manifold Theorem* which can be regarded as a profound generalization of the classical Bieberbach's Theorem.

Theorem 1.17 (Gromov 1978)**.** *For any $n \geq 2$, there exist constants $\epsilon = \epsilon(n) > 0$ and $w = w(n)$ such that if (M^n, g) satisfies*

$$\mathrm{diam}_g(M_j^n)^2 \cdot \max_{M^n} |\sec_{g_j}| \leq \epsilon, \tag{1.23}$$

then there exists a finite normal covering space $\widehat{M^n}$, of index bounded by $w(n)$, which is diffeomorphic to a nilmanifold.

Remark 1.3. *$\widehat{M^n}$ admits an iterated nilpotent fibration structure over a tori.*

Based on Gromov's characterization of almost flat manifolds, further studies of collapsing spaces became possible.

Theorem 1.18 (Fukaya 1987 and 1989)**.** *Let $(M_j^n, g_j) \xrightarrow{GH} (X_\infty^k, d_\infty)$ satisfy $k < n$ and $|\sec_{g_j}| \leq 1$.*

1. *If X_∞^k is a smooth, then there exists a fiber bundle map $N^{n-k} \to M_j^n \to X_\infty^k$ such that N^{n-k} is finitely covered by a nilmanifold.*
2. *In general, one has the following equivariant convergence diagram:*

$$\begin{tikzcd}
\widehat{N} \arrow[r] \arrow[d] & (F(M_j), O(n)) \arrow[r, "eqGH"] \arrow[d, "\pi_j"] & (Y_\infty, O(n)) \arrow[d, "\pi_\infty"] \\
N \arrow[r] & (M_j^n, g_j) \arrow[r, "GH"] & (X_\infty, d_\infty)
\end{tikzcd}$$

and there exists a singular fibration $N \to M_j \to X_\infty$ with N as its generic fiber, where Y_∞ is smooth, $\widehat{N}$ is a nilmanifold, and N is finitely covered by a nilmanifold.

Theorem 1.19 (Cheeger-Fukaya-Gromov 1992). *Let (M^n, g) be a complete Riemannian manifold. Then there exist $0 < \epsilon(n) < 1$ and $Q = Q(n)$ such that M^n admits a thick-thin decomposition $M^n = M^{\text{thick}}(\epsilon) \cup M^{\text{thin}}(\epsilon)$ which satisfies the following properties:*

(1) $\text{injrad}(x) \geq \epsilon > 0$ *for all $x \in M^{\text{thick}}(\epsilon)$. Therefore, $B_1(x)$ has finitely many diffeomorphism types of number bounded by Q, for any $x \in M^{\text{thick}}(\epsilon)$.*

(2) *$M^{\text{thin}}(\epsilon)$ has a locally finite covering $M^{\text{thin}}(\epsilon) \subset \bigcup_{\alpha\in\Lambda} \mathcal{U}_\alpha$ and for every $\alpha \in \Lambda$, there exists a singular fibration $\mathcal{N}_\alpha \to \mathcal{U}_\alpha \to X_\alpha$. If $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ and $\dim(\mathcal{U}_\alpha) > \dim(\mathcal{U}_\beta)$, then $\mathcal{U}_\alpha$ is nilpotent fiber bundle with fiber $\mathcal{U}_\beta$. Moreover, g is almost invariant in the following sense: there exists a smooth metric g_ϵ such that*

$$|\nabla^k(g - g_\epsilon)|_{C^0(M^{\text{thin}}(\epsilon))} < \epsilon \tag{1.24}$$

and g_ϵ is $\mathcal{N}_\alpha$-invariant for each $\alpha \in \Lambda$.

Theorem 1.20 (Shioya-Yamaguchi 2001 and 2005). *There exists an absolute constant $\epsilon > 0$ such that if (M^3, g) satisfies $\sec_g \geq -1$ and $\text{Vol}_g(B_1(x)) < v$, then M^3 is diffeomorphic to a graph manifold.*

2. Basic Comparison Geometry

The section is a crash course on basic comparison geometry of sectional and Ricci curvature. As applications, we will also prove some rigidity theorems.

2.1. Exponential map and comparison theorems.

Diameter and volume are common metric-geometric objects on a Riemannian manifold. These objects are defined via the metric structure of the manifold, whose properties are dominated by geodesic behaviors. Analytically, a geodesic, as a locally length-minimizing curve, yields a nonlinear ODE. Its linearization, called the Jacobi equation, is closely connected to the curvature of the underlying manifold.

Let (M^n, g) be a Riemannian manifold. Then let us recall that the exponential map $\mathrm{Exp}_p : T_pM^n \to M^n$ at a point $p \in M^n$ is defined as follows. For any $v \in T_pM^n$, let $\gamma : [0,1] \to M^n$ be the unique geodesic satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. Then $\mathrm{Exp}_p(v) \equiv \gamma(1)$.

Lemma 2.1. *In the above notations, the curve* $\mathrm{Exp}_p(sv)$ *always represents a geodesic starting from* p *with the initial vector* $v \in T_pM^n$*.*

Proof. Indeed, let $\gamma : [0,1] \to M^n$ be the unique geodesic satisfying $\gamma(0) = p$ and $\gamma'(0) = v$. For any $0 < s < 1$, letting $\gamma_s(t) = \gamma(st)$, we have $\gamma_s'(t) = s\gamma'$ which implies $\nabla_{\gamma_s'}\gamma_s' \equiv 0$. Checking the initial data, we have

$$\gamma_s(0) = p, \quad \gamma_s'(0) = sv. \tag{2.1}$$

Therefore, $\mathrm{Exp}_p(sv) = \gamma_s(1) = \gamma(s)$ which means Exp_p maps a line in T_pM^n to a geodesic. $\square$

Next, we will introduce the variation of a smooth curve. Let $U \subset M^n$ be a domain and let $\gamma : [a,b] \to U$ be a C^1-curve. A one-parameter variation of γ is a map

$$V : [a,b] \times (-1,1) \to U \tag{2.2}$$

such that $V(t,0) = \gamma(t)$ for all $t \in [a,b]$. From now on, we assume that V is piecewise C^k with $k \geq 1$. Note that there are two families of curves $\gamma_s(\cdot) \equiv V(\cdot, s)$ and $\sigma_t(\cdot) \equiv V(t, \cdot)$. The variation vector field $X(t)$ along γ is defined by

$$X(t) \equiv \frac{\partial}{\partial s}\Big|_{s=0} V(t,s) = \sigma_t'(0). \tag{2.3}$$

For the above family of curves γ_s, let us denote

$$E_s \equiv E_g(\gamma_s) = \int_a^b \langle \gamma_s', \gamma_s' \rangle dt, \tag{2.4}$$

for any $s \in (-1,1)$. Then we have the following first variation formula of arc length.

Lemma 2.2 (First variation of energy)**.** *Let $\gamma : [a,b] \to M^n$ be a smooth curve and let $V : [a,b] \times (-1,1) \to M^n$ be a smooth variation of γ. Let X be the variational vector field. Then the following holds,*

$$\frac{d}{ds}\Big|_{s=0} E_s = \int_a^b \langle \nabla_{\gamma'} X(t), \gamma'(t) \rangle dt = \langle X(b), \gamma'(b) \rangle - \langle X(a), \gamma'(a) \rangle - \int_a^b \langle X, \nabla_{\gamma'}\gamma' \rangle dt. \quad (2.5)$$

It is a fundamental fact in Riemannian geometry that the exponential map acts isometrically along the radial direction, which is called *Gauß Lemma*.

Lemma 2.3 (Gauß Lemma)**.** *For any $p \in M^n$, the exponential map Exp_p preserves the metric along radial directions, i.e., for any $t_0 \geq 0$,*

$$\langle D(\mathrm{Exp}_p)_{t_0 v}(v), D(\mathrm{Exp}_p)_{t_0 v}(w) \rangle = \langle v, w \rangle. \quad (2.6)$$

Proof. It suffices to consider the case when $w \perp v$. For simplicity, letting $\|v\| = \|w\| = 1$, we construct a family of lines

$$L(t,s) \equiv t(v \cos s + w \sin s) \quad (2.7)$$

and consider the variation $V(t,s) \equiv \mathrm{Exp}_p(L(t,s))$, $t \in [0, t_0]$, $s \in [-1,1]$ of $\gamma(t) \equiv \mathrm{Exp}_p(tv)$.

It follows from straightforward computations that $E_s = t_0$, and thus

$$0 = \frac{d}{ds}\Big|_{s=0} E_s = \langle X(t_0), D(\mathrm{Exp}_p)_{t_0 v}(v) \rangle - \langle w, v \rangle. \quad (2.8)$$

Since $X(t_0) = \frac{\partial}{\partial s}\Big|_{s=0} V(t_0, s) = t_0 \cdot D(\mathrm{Exp}_p)_{t_0 v}(w)$, we have that

$$\langle D(\mathrm{Exp}_p)_{t_0 v}(v), D(\mathrm{Exp}_p)_{t_0 v}(w) \rangle = 0, \quad (2.9)$$

which completes the proof. $\square$

Using the exponential map, let us make the following notation: given $p \in M^n$, the *conjugate domain* $\mathrm{Conj}(p) \subset T_p M^n$ is the set of vectors $v \in T_p M^n$ such that Exp_p is not singular at tv for any $0 < t \leq 1$. As a comparison, let $\mathrm{Seg}(p) \subset T_p M^n$ be the set of vectors $v \in T_p M^n$ such that

$$d_g(p, \mathrm{Exp}_p(v)) = \|v\|, \quad (2.10)$$

called the *segment domain* of Exp_p. We also define

$$\mathrm{injrad}_p \equiv \sup\{r > 0 : B_r(0^n) \subset T_p M^n \cap \mathrm{Seg}(p)\}, \quad (2.11)$$

$$\mathrm{conjrad}_p \equiv \sup\{r > 0 : B_r(0^n) \subset T_p M^n \cap \mathrm{Conj}(p)\}. \quad (2.12)$$

Definition 2.1 (Cut locus)**.** *Denote by*

$$\ell_{x,v} \equiv \max\{t_0 > 0 | \gamma_{x,v} : [0,t] \to M^n \textit{ is minimizing for each } t < t_0\}, \quad (2.13)$$

then for $x \in M^n$ the cut locus $\mathcal{C}_x$ is defined by

$$\mathcal{C}_x \equiv \{y \in M^n | y \equiv \gamma_{x,v}(\ell_{x,v}) \textit{ for some } x, v\}. \quad (2.14)$$

Given $x \in M^n$, consider the ϵ-tubular neighborhood of $\mathcal{C}_x$ defined by

$$\mathcal{C}_{x,\epsilon} \equiv \{y \in M^n | y \equiv \gamma_{x,v}(\ell_{x,v} - t),\ |t| < \epsilon\}. \tag{2.15}$$

Notice that, we can take a smooth neighborhood $\mathcal{C}_x \subset U_\epsilon \subset C_{x,\epsilon}$ such that if N is the outward normal of $\partial(B_R(x) \setminus U_\epsilon)$, then

$$\langle N, \nabla d_x \rangle > 0. \tag{2.16}$$

Now let $p \in M^n$ and let $r(x) \equiv d_g(x, p)$ be the distance to p. One can choose a geodesic polar coordinate $\{r, \theta_1, \ldots, \theta_{n-1}\}$ via the exponential map. Then it follows from Gauß Lemma that $|\partial_r| = |\nabla r| = 1$.

In studying local geometry of Riemannian manifolds, the most important coordinate system is the *geodesic normal coordinates*.

Lemma 2.4 (Geodesic normal coordinates)**.** *Let (M^n, g) be a Riemannian manifold. For any $p \in M^n$, there is a coordinate system $\{x_i\}_{i=1}^n$ such that $g_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$ for all $1 \leq i, j, k \leq n$. In particular, as $r \to 0$,*

$$|g_{ij} - \delta_{ij}| = O(r^2) \quad \text{and} \quad |\nabla - \nabla^0| = O(r). \tag{2.17}$$

With the above preparations, we are ready to connect the behaviors of geodesics and curvature. A primary difficulty is that geodesics yield nonlinear differential equations. Thus one needs first understand its linearization, i.e., *geodesic variation*.

On a Riemannian manifold (M^n, g), given a curve $\gamma : [a, b] \to M^n$, a variation

$$V : [a, b] \times (-1, 1) \to M^n \tag{2.18}$$

of γ, by definition, is a smooth surface with $V(t, 0) = \gamma(t)$. Now a variation $V : [a, b] \times (-1, 1) \to M^n$ of γ is called a geodesic variation if $\gamma_s(\cdot) \equiv V(\cdot, s)$ is a geodesic for any $s \in (-1, 1)$. The variation vector field

$$J(t) \equiv \frac{\partial}{\partial s}\Big|_{s=0} V(t, s) \tag{2.19}$$

is called a *Jacobi field along* γ. Let $p \in U \subset \mathbb{R}^n$ and we consider a specific geodesic variation

$$V(t, s) = \mathrm{Exp}_p(t(u + sv)). \tag{2.20}$$

Then its variation vector field $J(t)$ is expressed as $J(t) = t \cdot (D\,\mathrm{Exp})_{tu}(v)$. Obviously, $J(0) = 0$ and $J'(0) = v$. As a linearization, a Jacobi field satisfies a linear ODE

Lemma 2.5. *Let $J(t)$ be a Jacobi field along a geodesic γ. Then J satisfies*

$$J''(t) + \mathrm{Rm}(J(t), \gamma'(t))\gamma'(t) = 0. \tag{2.21}$$

Proof. Let $V : [a, b] \times (-1, 1) \to M^n$ be a geodesic variation of a geodesic γ. We denote

$$\mathcal{J} \equiv V_*\left(\frac{\partial}{\partial s}\right), \quad \mathcal{T} \equiv V_*\left(\frac{\partial}{\partial t}\right). \tag{2.22}$$

So it follows that

$$\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\mathcal{J} = \nabla_{\mathcal{T}}\nabla_{\mathcal{J}}\mathcal{T} - \nabla_{\mathcal{J}}\nabla_{\mathcal{T}}\mathcal{T} + \nabla_{\mathcal{J}}\nabla_{\mathcal{T}}\mathcal{T}. \tag{2.23}$$

Since V is a geodesic variation, $\nabla_{\mathcal{T}}\mathcal{T} \equiv 0$. Therefore,

$$\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\mathcal{J} = \nabla_{\mathcal{T}}\nabla_{\mathcal{J}}\mathcal{T} - \nabla_{\mathcal{J}}\nabla_{\mathcal{T}}\mathcal{T} = -\operatorname{Rm}(\mathcal{J}, \mathcal{T})\mathcal{T}. \tag{2.24}$$

Letting $s = 0$, the desired equation follows. □

Let $\gamma : [0, 1] \to U$ be the geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Let $U(t)$ and $W(t)$ be the Jacobi fields along a geodesic γ that determined by the initial values $U(0) = W(0) = 0$, $U'(0) = u$, and $W'(0) = w$. Let us compute the Taylor expansion of the 1-variable function $g(U(t), W(t))$ at $t = 0$. To do this, the coefficients at 0 are given by

$$g(U(0), W(0)) = 0, \quad (g(U, W))'(0) = 0, \quad (g(U, W))''(0) = 2g(u, w), \tag{2.25}$$

$$(g(U, W))'''(0) = 0, \quad (g(U, W))^{(4)}(0) = -8R(v, u, w, v). \tag{2.26}$$

Let U be the Jacobi field along γ with $U(0) = 0$ and $U'(0) = u$. Then the Taylor expansion of $|U(t)|^2$ along γ at $t = 0$ is given by

$$|J(t)|^2 = t^2 - \frac{1}{3}\sec(v, u)t^4 + O(|t|^5). \tag{2.27}$$

Using the above calculations, we are able to compute the Taylor expansion of (g_{ij}) along a geodesic γ. To do this, now let us consider the geodesic variation in geodesic normal coordinates $\{x_i\}_{i=1}^n$,

$$V_i(t, s) \equiv (tv_1, \ldots, t(v_i + s), tv_n). \tag{2.28}$$

Then the Jacobi field $J_i(t)$ can be expressed as $J_i(t) = t\partial_i$ which implies that

$$g_{ij} = g(\partial_i, \partial_j) = t^{-2}g(J_i(t), J_j(t)). \tag{2.29}$$

Therefore,

$$g_{ij} = \delta_{ij} - \frac{R(x, e_i, e_j, x)}{3}t^2 + O(|x|^3) = \delta_{ij} - \frac{1}{3}R_{kij\ell}x^k x^\ell + O(|x|^3), \tag{2.30}$$

where $x^k \equiv t \cdot v_k$. Similarly, one has the following expansions

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6}\operatorname{Ric}_{k\ell}(p)x^k x^\ell + O(|x|^3), \tag{2.31}$$

$$\frac{\operatorname{Vol}_g(B_r(p))}{\operatorname{Vol}_0(B_r)} = 1 - \frac{\operatorname{Scal}_g}{6(n+2)}r^2 + O(r^4). \tag{2.32}$$

Riemann curvature also relates the metric or geodesic behavior via the Hessian of the distance function. *The shape operator* $S : T_pM^n \to T_pM^n$ is defined as

$$S(X) \equiv \nabla_X \partial_r. \tag{2.33}$$

Immediately, one can see that the shape operator is related to the Hessian of r by

$$\langle S(X), Y\rangle = \nabla^2 r(X,Y) = \langle \nabla_X \nabla r, Y\rangle. \tag{2.34}$$

Lemma 2.6. *Let (M^n, g) be a Riemannian manifold. Then following identities hold.*

(1) *(Radial variation) $\mathcal{L}_{\partial_r} g = 2\nabla^2 r$.*
(2) *(Ricatti equation) Let $R_{\partial_r}(X) \equiv \mathrm{Rm}(X, \partial_r)\partial_r$. Then $\nabla_{\partial_r} S + S^2 + R_{\partial_r} = 0$.*

Theorem 2.7 (Metric and Hessian Comparison)**.** *Let (M^n, g) satisfy $\kappa \le \sec_g \le K$. Then for any $p \in M^n$, the following holds on $\partial B_r(p)$ for any $r \in (0, \mathrm{injrad}_p)$:*

(1) *(metric comparison) $g_K(r) \le g(r) \le g_\kappa(r)$;*
(2) *(Hessian comparison) $S_K \le S(r) \le S_\kappa(r)$;*
(3) *(local behavior of Hessian) $S(r) = \frac{1}{r} I + O(r),\ r \to 0$.*

Here $g_k(r) \equiv \mathrm{sn}_k(r) I$, $S_k(r) \equiv \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} I$, and

$$\mathrm{sn}_k(r) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k} r), & k > 0, \\ r, & k = 0, \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k} r), & k < 0. \end{cases} \tag{2.35}$$

Remark 2.1. *Given $\kappa \in \mathbb{R}$, the function $\mathrm{sn}_k(r)$ is the unique solution of the ODE*

$$f''(r) + \kappa f(r) = 0, \quad f(0) = 0, \quad f'(0) = 1. \tag{2.36}$$

The above metric and Hessian comparisons give a unified proof of the Jacobi field comparison.

Theorem 2.8 (Rauch/Berger)**.** *Let (M^n, g) be complete and $\kappa \le sec_g \le K$. Let $\gamma(t)$ be a geodesic without conjugate points. If a Jacobi field $J(t)$ along $\gamma(t)$ that satisfies the initial data*

$$\begin{cases} J(0) = J_\kappa(0) = J_K(0) = 0 \\ |J'(0)| = |J_\kappa'(0)| = |J_K'(0)| \ne 0 \\ \langle J'(0), \gamma'(0)\rangle = \langle J_\kappa'(0), \gamma'(0)\rangle_{g_\kappa} \\ = \langle J_K'(0), \gamma'(0)\rangle_{g_K} \end{cases} \quad or \quad \begin{cases} J'(0) = J_\kappa'(0) = J_K'(0) = 0 \\ |J(0)| = |J_\kappa(0)| = |J_K(0)| \ne 0 \\ \langle J(0), \gamma'(0)\rangle = \langle J_\kappa(0), \gamma'(0)\rangle_{g_\kappa} \\ = \langle J_K(0), \gamma'(0)\rangle_{g_K}, \end{cases} \tag{2.37}$$

then

$$|J_K(t)| \le |J(t)| \le |J_\kappa(t)|. \tag{2.38}$$

Ricci curvature, by definition, is the trace of Riemann curvature tensor. Since sectional curvature directly controls the behaviors of the metric tensor and the Hessian of the distance function, it is expected that the comparison geometry of Ricci curvature gives an "averaged" control on the Riemannian metrics, namely volume comparison and Laplacian comparison.

In technical computations, Bochner formula plays a fundamental role.

Lemma 2.9 (Bochner)**.** *Let $f \in C^\infty(M^n)$. Then*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f). \tag{2.39}$$

Corollary 2.10 (Riccati equation)**.** *Let $p \in M^n$ and let $r(x) \equiv d(p,x)$ be the distance function to p. Then the following holds:*

$$\partial_r H + |\,\mathrm{II}\,|^2 + \mathrm{Ric}_{rr} = 0. \tag{2.40}$$

Theorem 2.11 (Local comparison)**.** *Let (M^n, g) be complete with $\mathrm{Ric}_g \geq (n-1)\kappa$. Given $p \in M^n$, let $r(x) \equiv d(p,x)$ be the distance function to p. Then the following holds:*

(1) *(Laplacian) $\Delta r \leq (n-1)\frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)}$ when r is smooth.*
(2) *(Volume density) $\sqrt{G} \leq \mathrm{sn}_k^{n-1}(r)$. Moreover, "=" holds for all $r > 0$ iff M has constant curvature κ.*

Lemma 2.12. *If a C^1-function $u(r)$ satisfies*

$$\begin{cases} -K \leq u'(r) + u^2(r) \leq -\kappa \\ u(r) = \frac{1}{r} + O(r) \text{ as } r \to 0, \end{cases} \tag{2.41}$$

or

$$\begin{cases} -K \leq u'(r) + u^2(r) \leq -k \\ u(0) = 0. \end{cases} \tag{2.42}$$

Then $u_K(r) \leq u(r) \leq u_\kappa(r)$, where $u_k(r) = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$ for any $k \in \mathbb{R}$.

Proof of Theorem 2.11. Proof of item (1). First, we will prove that as $r \to 0$,

$$H(r) = (n-1)\frac{1}{r} + O(r). \tag{2.43}$$

Let us denote by g_0 the Euclidean metric on $\mathrm{Exp}_p(\mathring{\mathrm{Seg}})$. Since the Taylor expansion of the metric tensor g_{ij} (in normal coordinates) along γ is given by

$$g_{ij}(t) = g_0 + O(t^2), \tag{2.44}$$

applying Koszul's formula, we have

$$\nabla = \nabla^0 + O(r), \tag{2.45}$$

where $\nabla = \nabla^g$ and $\nabla^0 = \nabla^{g_0}$. Therefore,

$$\mathrm{II}_{\partial_r} = \nabla(\nabla r) = \nabla(\nabla^0 r_0) = (\nabla^0 + O(r))(\nabla^0(r_0)) = (\nabla^0)^2 r_0 + O(r) = \frac{1}{r}\cdot g_0 + O(r). \tag{2.46}$$

Tracing the above expansion, we have $H(r) = \frac{n-1}{r} + O(r)$. Let $u(r) = \frac{H(r)}{n-1}$. Applying Lemma 2.12 to $u(r)$, the Laplacian comparison just follows.

Now we prove item (2). Let us denote $G \equiv \det(g)$ and $\mu \equiv \sqrt{G}$. Then it follows that

$$\begin{aligned}\Big(\frac{\mu}{\mu_\kappa}\Big)' &= \frac{\mu'(r)\mu_\kappa(r) - \mu(r)\mu_\kappa'(r)}{\mu_\kappa^2} \\ &= \frac{(H(r) - H_\kappa(r))\mu(r)\mu_\kappa(r)}{\mu_\kappa^2} \\ &\leq 0. \end{aligned}\tag{2.47}$$

Now assuming $\mu(r) = \mu_\kappa(r)$ for all $r \geq 0$. Then $H(r)\mu(r) = \mu'(r) = \mu_\kappa'(r) = H_\kappa(r)\mu_\kappa(r)$, and hence $H(r) = m_\kappa(r) = (n-1)\frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)}$, and $\mu(r) = \mathrm{sn}_\kappa^{n-1}(r) I$. Therefore,

$$-(n-1)\kappa = H' + \frac{H^2}{n-1} \leq H' + \mathrm{Tr}(\mathrm{II}^2) = -\operatorname{Ric}_{rr} \leq -(n-1)\kappa. \tag{2.48}$$

"=" holds in all the above inequalities, which implies $(\Delta r)^2 = (n-1)\,\mathrm{Tr}(\mathrm{II}^2)$. Equivalently, $\lambda_1 = \ldots = \lambda_n = \lambda = \frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)}$, where λ_i's are $(n-1)$ non-zero eigenvalues of $\nabla^2 r$. Consequently,

$$\nabla^2 r = \lambda \cdot \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{n-1}\end{pmatrix}, \tag{2.49}$$

and hence $G(r) = G_\kappa(r)$. It follows from the fact $\mathfrak{L}_{\partial_r} g = 2\,\mathrm{Hess}(r) g$, and the initial condition $g_{ri}(0) = g_{ij}(0) = 0$ $(i \neq j)$, $g_{rr}(0) = g_{ii}(0) = 1$ that $g_{ij} = g_{ir} = 0$ $(i \neq j)$, $g_{rr} = 1$, and $g_{ii}(r) = \mathrm{sn}_\kappa^2(r)$. Therefore, g has constant curvature κ. □

Theorem 2.13 (Global Laplcian Comparison I). *Let (M^n, g) be complete with* $\mathrm{Ric} \geq (n-1)\kappa$. *Let $r(x) \equiv d(p,x)$ for $p \in M^n$. Then Δr is a signed Radon measure on (M^n, g) with the decomposition*

$$\Delta r = \mu_{ac} + \mu_{sing} \tag{2.50}$$

such that

$$\mu_{ac} \leq (n-1)\frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)} \tag{2.51}$$

and μ_{sing} is non-positive and supported in $\mathcal{C}_p$, where $\mathcal{C}_p$ denotes the cut locus of p. In particular, the Laplacian comparison holds in the distributional sense, namely for any $\varphi \in C_0^\infty(M^n)$, $\varphi \geq 0$, we have

$$\int_{M^n} r\Delta\varphi \,\mathrm{dvol}_g \leq \int_{M^n} \Big((n-1)\frac{\mathrm{sn}_\kappa'(r)}{\mathrm{sn}_\kappa(r)}\Big)\cdot\varphi \,\mathrm{dvol}_g\,. \tag{2.52}$$

Proof. We only verify that μ_{sing} is non-positive. Let $B_R+(y) \subset B_R(y)\setminus(\{p\}\cup\mathcal{C}_p)$ be the set on which $\Delta r>0$. Let $\partial B_R^-(y)\subset\partial B_R(y)\setminus(\{p\}\cup\mathcal{C}_p)$ denote the subset on which $\langle\nabla r, N\rangle<0$, where N is the outward unit normal vector field. For any $\epsilon>0$, taking some open set U_ϵ such that $\mathcal{C}_p\subset U_\epsilon\subset T_\epsilon(\mathcal{C}_p)$. Let $\widehat{N}$ be the outward unit normal vector field to $\partial(B_R(y)\setminus U_\epsilon)$ at $x\in B_R(y)\cap\partial U_\epsilon$. Then $\langle\nabla r,\widehat{N}\rangle\geq 0$.

Let $f\in C^\infty(M^n)$ with $\nabla f\equiv 0$ near $\partial B_R(y)$. Then

$$\begin{aligned}\lim_{\eta\to 0}\int_{B_R(y)\setminus B_\eta(q)} r\Delta f &= -\int_{B_R(y)}\langle\nabla r,\nabla f\rangle\\ &=\lim_{\epsilon\to 0}\int_{B_R(y)\setminus U_\epsilon} f\Delta r-\lim_{\epsilon\to 0}\int_{\partial B_R(y)\setminus U_\epsilon}\langle\nabla r, N\rangle f-\lim_{\epsilon\to 0}\int_{B_R(y)\setminus\partial U_\epsilon}\langle\nabla r,N\rangle f\\ &=\int_{B_R(y)\setminus\mathcal{C}_p} f\Delta r-\int_{\partial B_R(y)}\langle\nabla r,N\rangle f-\lim_{\epsilon\to 0}\int_{B_R(y)\cap\partial U_\epsilon}\langle\nabla r,\widehat{N}\rangle f. \quad (2.53)\end{aligned}$$

Notice that the last term in the above equality is *non-positive*. Letting $f\equiv 1$, the conclusion follows. ☐

2.2. **Volume comparison and applications.**

Theorem 2.14 (General relative volume comparison)**.** *Let (M^n,g,p) be complete such that* $\mathrm{Ric}\geq(n-1)\kappa$. *Assume that $r_1\leq r_2\leq r_3\leq r_4$, then*

$$\frac{\mathrm{Vol}(A_{r_3,r_4}(p))}{\mathrm{Vol}_k(A_{r_3,r_4}(0^*))}\leq\frac{\mathrm{Vol}(A_{r_1,r_2}(p))}{\mathrm{Vol}_k(A_{r_1,r_2}(0^*))}. \quad (2.54)$$

Equality holds if and only if $A_{r_1,r_4}(p)$ is isometric to $A_{r_1,r_4}(0^*)\subset S^n_\kappa$.

Proof. Let $f(r)\equiv\frac{\mathrm{Area}(\partial B_r(x))}{\mathrm{Area}_\kappa(\partial B_r(0^*))}=\frac{\mathrm{Area}(\partial B_r(x))}{\sqrt{G_\kappa(r)}\cdot\omega_{n-1}}$. Let $\{r,\Theta\}$ be the geodesic polar coordinate system. Then

$$\begin{aligned}\frac{df}{dr}&=\frac{1}{\omega_{n-1}}\int_{S^{n-1}}\frac{\sqrt{G}}{\sqrt{G_\kappa}}\cdot\Big(\frac{\partial_r\sqrt{G}}{\sqrt{G}}-\frac{\partial_r\sqrt{G_\kappa}}{\sqrt{G_\kappa}}\Big)d\Theta\\ &\leq\frac{1}{\omega_{n-1}}\int_{S^{n-1}}\frac{\sqrt{G}}{\sqrt{G_\kappa}}\cdot(H(r)-H_\kappa(r))d\Theta\leq 0, \quad (2.55)\end{aligned}$$

and $f(0) = 1$. Next,

$$\begin{aligned}
\mathrm{Vol}(A_{r_3,r_4}(x)) \cdot \mathrm{Vol}_\kappa(A_{r_1,r_2}(0^*)) &= \Big(\int_{r_3}^{r_4} A(t)dt\Big) \cdot \Big(\int_{r_1}^{r_2} A_\kappa(t)dt\Big) \\
&= \Big(\int_{r_3}^{r_4} f(t)A_\kappa(t)\Big) \cdot \Big(\int_{r_1}^{r_2} A_\kappa(t)dt\Big) \\
&\leq f(r_3)\Big(\int_{r_3}^{r_4} A_\kappa(t)dt\Big) \cdot \Big(\int_{r_1}^{r_2} A_\kappa(t)dt\Big) \\
&\leq \Big(\int_{r_3}^{r_4} A_\kappa(t)dt\Big) \cdot \Big(\int_{r_1}^{r_2} f(t)A_\kappa(t)dt\Big) \\
&= \mathrm{Vol}_\kappa(A_{r_3,r_4}(0^*)) \cdot \mathrm{Vol}(A_{r_1,r_2}(x)). \qquad (2.56)
\end{aligned}$$

So the proof of the comparison is done.

Next, when equality holds, $f(r_1) = f(r_4)$. Then the proof of Laplacian comparison implies that $A_{r_1,r_4}(p)$ is isometric to $A_{r_1,r_4}(0^n)$. □

Corollary 2.15. *Let (M^n, g, p) be complete such that* $\mathrm{Ric} \geq (n-1)\kappa$. *For any* $R > 0$,

$$\frac{\mathrm{Area}(\partial B_R(p))}{\mathrm{Area}_k(\partial B_R(0^*))} \leq \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^*))}, \qquad (2.57)$$

and equality holds if and only if $B_R(p)$ is isometric to $B_R(0^*) \subset S^n_k$.

Proof. For any $\epsilon > 0$ and $R > 0$, by Theorem 2.14,

$$\frac{\mathrm{Vol}(B_{R+\epsilon}(p)) - \mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_{R+\epsilon}(0^*)) - \mathrm{Vol}_\kappa(B_R(0^*))} = \frac{\mathrm{Vol}(A_{R,R+\epsilon}(p))}{\mathrm{Vol}_\kappa(A_{R,R+\epsilon}(0^*))} \leq \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))}, \qquad (2.58)$$

which implies

$$\frac{\frac{1}{\epsilon} \cdot (\mathrm{Vol}(B_{R+\epsilon}(p)) - \mathrm{Vol}(B_R(p)))}{\frac{1}{\epsilon} \cdot (\mathrm{Vol}_\kappa(B_{R+\epsilon}(0^*)) - \mathrm{Vol}_\kappa(B_R(0^*)))} \leq \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))}. \qquad (2.59)$$

Letting $\epsilon \to 0$, the desired comparison immediately follows.

For the rigidity part, we assume $\frac{\mathrm{Area}(\partial B_R(p))}{\mathrm{Area}_\kappa(\partial B_R(0^*))} = \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))}$ and let

$$V(r) \equiv \mathrm{Vol}(B_r(p)), \quad V_\kappa(r) \equiv \mathrm{Vol}_\kappa(B_r(0^*)) \qquad (2.60)$$

and

$$S(r) \equiv \mathrm{Area}(\partial B_r(p)), \quad S_\kappa(r) \equiv \mathrm{Area}_\kappa(\partial B_r(0^*)). \qquad (2.61)$$

Then

$$\frac{V(R)}{S(R)} = \frac{\int_0^R S(t)dt}{S(R)} = \int_0^R \frac{S(t)}{S(R)}dt. \qquad (2.62)$$

By the relative area comparison, it holds that for every $0 < t \leq R$,

$$\frac{S(t)}{S(R)} \geq \frac{S_\kappa(t)}{S_\kappa(R)}. \qquad (2.63)$$

Therefore,

$$\frac{V(R)}{S(R)} \geq \int_0^R \frac{S_\kappa(t)}{S_\kappa(R)} = \frac{\int_0^R S_\kappa(t)dt}{S_\kappa(R)} = \frac{V_\kappa(R)}{S_\kappa(R)}. \tag{2.64}$$

The assumption $\frac{V(R)}{S(R)} = \frac{V_\kappa(R)}{S_\kappa(R)}$ and the above inequality imply that for every $0 < t \leq R$,

$$\frac{S(R)}{S_\kappa(R)} \equiv \frac{S(t)}{S_\kappa(t)}. \tag{2.65}$$

Since $\lim_{t\to 0} \frac{S(t)}{S_\kappa(t)} = 1$, so for every $0 < t < R$, $S(t) = S_\kappa(t)$ and hence $V(t) = V_\kappa(t)$. Then $B_R(p)$ is isometric to $B_R(0^n) \subset S^n_\kappa$.

□

Theorem 2.16 (Bishop-Gromov's relative volume comparison)**.** *Let (M^n, g) be complete and satisfy* $\mathrm{Ric}_g \geq (n-1)\kappa$*. Then for any $x \in M^n$, the quantity*

$$Q_x(r) \equiv \frac{\mathrm{Vol}(B_r(p))}{V_k(r)} \tag{2.66}$$

is monotone decreasing with $\lim_{r\to 0} Q_x(x) = 1$, where $V_k(r) \equiv \mathrm{Vol}_\kappa(B_r(0^*))$ and $B_r(0^*) \subset S^n_\kappa$. Moreover, if $Q_x(R) = Q_x(r)$ for some $r \leq R$, then $B_R(x)$ is isometric to $B_R(0^*)$.

Proof. For any $R > r > 0$, applying Theorem 2.14,

$$\frac{\mathrm{Vol}(B_R(p)) - \mathrm{Vol}(B_r(p))}{\mathrm{Vol}_\kappa(B_R(0^*)) - \mathrm{Vol}_\kappa(B_r(0^*))} \leq \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_\kappa(B_r(0^*))}. \tag{2.67}$$

Straightforward computations imply

$$\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_\kappa(B_R(0^*))} \leq \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_\kappa(B_r(0^*))}. \tag{2.68}$$

Now let us prove the rigidity part. First, by volume comparison, for any $\epsilon \in (0, R-r)$,

$$\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))} \leq \frac{\mathrm{Vol}(B_{R-\epsilon}(p))}{\mathrm{Vol}_k(B_{R-\epsilon}(0^n))} \leq \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}. \tag{2.69}$$

If $\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}$ holds for some $0 < r < R$, we have that for every $\epsilon \in (0, R-r)$,

$$\frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))} = \frac{\mathrm{Vol}(B_{R-\epsilon}(p))}{\mathrm{Vol}_k(B_{R-\epsilon}(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}, \tag{2.70}$$

and hence for every $\epsilon > 0$

$$\frac{\mathrm{Vol}(B_R(p)) - \mathrm{Vol}(B_{R-\epsilon}(p))}{\mathrm{Vol}_k(B_R(0^n)) - \mathrm{Vol}_k(B_{R-\epsilon}(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))}. \tag{2.71}$$

Let $\epsilon \to 0$,

$$\frac{\mathrm{Vol}(\partial B_R(p))}{\mathrm{Vol}_k(\partial B_R(0^n))} = \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_k(B_r(0^n))} = \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}_k(B_R(0^n))}. \tag{2.72}$$

Then the isometric rigidity follows from Corollary 2.15. □

Theorem 2.17 (Bonnet-Myers)**.** *Let* (M^n, g) *satisfy* $\mathrm{Ric}_g \geq n-1$. *Then* $\mathrm{diam}_g(M^n) \leq \pi$.

Proof. We prove it by contradiction. Suppose there exists $\epsilon > 0$ such that $\mathrm{diam}_g(M^n) = \pi+\epsilon$. Let $\gamma : [0, \pi+\epsilon] \to M^n$ be a minimal geodesic connecting $p, q \in M^n$ such that $L(\gamma) = \pi + \epsilon$.

We denote $r(x) \equiv d(x, p)$. Then r is smooth at $\gamma(t)$ for $t \in (0, \pi + \epsilon)$. By Laplacian comparison,

$$(\Delta r)(\gamma(t)) \leq (n-1) \cdot \frac{\cos t}{\sin t} \to -\infty, \text{ as } t \to \pi. \tag{2.73}$$

Contradiction. $\square$

Theorem 2.18 (Cheng's Maximal Diameter Theorem)**.** *Let* (M^n, g) *satisfy* $\mathrm{Ric}_g \geq n-1$ *and* $\mathrm{diam}_g(M^n) = \pi$. *Then* (M^n, g) *must be isometric to the round sphere* $\mathbb{S}^n$ *of curvature* $+1$.

Proof. We will apply the relative volume comparison theorem to prove this rigidity. Let $p, q \in M^n$ satisfy $d_g(p, q) = \pi$ Let us denote $B_{\frac{\pi}{2}}(0^*) \equiv \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_{n+1} > 0\}$. Notice that

$$B_\pi(p) = B_\pi(q) = M^n, \tag{2.74}$$

and hence

$$\mathrm{Vol}_g(B_\pi(p)) = \mathrm{Vol}_g(B_\pi(q)) = \mathrm{Vol}_g(M^n). \tag{2.75}$$

Applying volume comparison,

$$\frac{\mathrm{Vol}(B_{\frac{\pi}{2}}(p))}{\mathrm{Vol}_{g_1}(B_{\frac{\pi}{2}}(0^*))} \geq \frac{\mathrm{Vol}(B_\pi(p))}{\mathrm{Vol}(\mathbb{S}^n)}. \tag{2.76}$$

Immediately,

$$\frac{\mathrm{Vol}(B_{\frac{\pi}{2}}(p))}{\mathrm{Vol}(M^n)} \geq \frac{\mathrm{Vol}_{g_1}(B_{\frac{\pi}{2}}(0^*))}{\mathrm{Vol}(\mathbb{S}^n)} = \frac{1}{2}. \tag{2.77}$$

"=' holds iff $B_\pi(p)$ is isometric to $\mathbb{S}^n$. Similarly, $\mathrm{Vol}(B_{\frac{\pi}{2}}(q)) \geq \frac{1}{2}\mathrm{Vol}(M^n)$. On the other hand, $\dot{(}p, q) = \pi$ implies $B_{\frac{\pi}{2}}(p) = B_{\frac{\pi}{2}}(q) = \emptyset$. Therefore,

$$\mathrm{Vol}(M^n) \geq \mathrm{Vol}(B_{\frac{\pi}{2}}(p)) + \mathrm{Vol}(B_{\frac{\pi}{2}}(q)) \geq \frac{1}{2}\mathrm{Vol}(M^n) + \frac{1}{2}\mathrm{Vol}(M^n) = \mathrm{Vol}(M^n). \tag{2.78}$$

Therefore,

$$\mathrm{Vol}_g(B_{\frac{\pi}{2}}(p)) = \mathrm{Vol}_g(B_{\frac{\pi}{2}}(q)) = \frac{1}{2}\mathrm{Vol}(M^n). \tag{2.79}$$

Applying Bishop-Gromov's volume comparison, M^n is isometric to $\mathbb{S}^n$. $\square$

3. Geometry of metric spaces: Gromov-Hausdorff theory

This section introduces basic concept of Gromov-Hausdorff distance between metric spaces, which is the foundation of studying the metric aspect of Riemannian geometry. We will also present several basic examples.

3.1. Space of metric spaces and Gromov's precompactness theorem.

Definition 3.1 (Hausdorff distance). *Let (Z, d) be a metric space. Let $A, B \subset Z$ be compact. Then*

$$d_H(A, B) \equiv \inf\{r > 0 | B \subset T_r(A) \text{ and } A \subset T_r(B)\}. \tag{3.1}$$

Theorem 3.1. *Denote by $\mathfrak{M}(Z)$ the collection of all subsets in the metric space (Z, d). Then $(\mathfrak{M}(Z), d_H)$ is a metric space.*

Theorem 3.2. *If (Z, d) is compact, then $(\mathfrak{M}(Z), d_H)$ is also compact.*

Definition 3.2 (Gromov-Hausdorff distance). *Let (X, d_X) and (Y, d_Y) be compact metric spaces. Then we define*

$$d_{GH}(X, Y) \equiv \inf_{Z,\phi,\psi} \left\{ d_H^Z(\phi(X), \psi(Y)) : \phi : X \to Z,\ \psi : Y \to Z \text{ are isometric embeddings} \right\}.$$

Let A and B be two sets. Then we define

$$A \sqcup B \equiv \{(x, 0) : x \in A\} \cup \{(y, 1) : y \in B\}. \tag{3.2}$$

Let (X, d_X) and (Y, d_Y) be metric spaces. We denote by $\bar{d}$ the *admissible metric* on $X \sqcup Y$ which isometrically extends d_X and d_Y into $X \sqcup Y$.

Lemma 3.3. *Let (X, d_X) and (Y, d_Y) be compact metric spaces. We define*

$$\hat{d}_{GH}(X, Y) \equiv \inf_{\bar{d}} \left\{ d_H(X, Y) : \bar{d} \text{ is an admissible metric on } X \sqcup Y \right\}. \tag{3.3}$$

Then $d_{GH} = \hat{d}_{GH}$.

Proof. By definition, $d_{GH} \le \hat{d}_{GH}$. We will prove $\hat{d}_{GH} \le d_{GH}$. For any $\epsilon > 0$, there exist a metric space (Z, d) and isometric embeddings

$$\phi : X \hookrightarrow Z, \quad \psi : Y \hookrightarrow Z \tag{3.4}$$

such that $d_H(\phi(X), \psi(Y)) \le d_{GH}(X, Y) + \epsilon$. Let us consider the product metric d^ϵ on $Z \times [0, \epsilon]$ and isometric embeddings

$$\phi_0 \equiv (\phi, 0) : X \times \{0\} \hookrightarrow Z \times \{0\}, \quad \psi_\epsilon \equiv (\psi, \epsilon) : Y \times \{\epsilon\} \hookrightarrow Z \times \{\epsilon\}. \tag{3.5}$$

The restriction of d^ϵ onto $(\phi(X)\times\{0\})\cup(\psi(Y)\times\{\epsilon\})\subset Z\times[0,\epsilon]$ gives an admissible metric $\bar{d}^\epsilon$ on $X\sqcup Y$ (realized by $\phi_0(X)\sqcup\psi_\epsilon(Y)$). Then

$$\begin{aligned}\hat{d}_{GH}(X,Y) &\le \bar{d}^\epsilon_H(X,Y)\\ &= \bar{d}^\epsilon_H(\phi_0(X),\psi_\epsilon(Y))\\ &\le \bar{d}^\epsilon_H(\phi_0(X),\phi(X)\times\{\epsilon\})+\bar{d}^\epsilon_H(\phi(X)\times\{\epsilon\},\psi(Y)\times\{\epsilon\})\\ &\le 2\epsilon+d_{GH}(X,Y), \qquad (3.6)\end{aligned}$$

which completes the proof. □

Lemma 3.4. *Denote by $\mathcal{M}et$ the collection of all compact metric spaces. Then d_{GH} is a pseudo metric on $\mathcal{M}et$. Furthermore, $d_{GH}(X,Y)=0$ if and only if X is isometric to Y.*

Proof. First, let us prove the triangle inequality. Given any compact metric spaces (X,d_X), (Y,d_Y) and (Z,d_Z), we will show that for any $\epsilon>0$,

$$d_{GH}(X,Z)\le d_{GH}(X,Y)+d_{GH}(Y,Z)+\epsilon. \qquad (3.7)$$

Taking admissible metrics d_{XY} and d_{YZ} on $X\sqcup Y$ and $Y\sqcup Z$, respectively, such that

$$d_{XY,H}(X,Y)\le d_{GH}(X,Y)+\frac{\epsilon}{2},\quad d_{YZ,H}(Y,Z)\le d_{GH}(Y,Z)+\frac{\epsilon}{2}. \qquad (3.8)$$

One can check that

$$d_{XZ}(x,z)\equiv\inf\{d_{XY}(x,y)+d_{YZ}(y,z):y\in Y\} \qquad (3.9)$$

is an admissible metric on $X\sqcup Z$. Then the collection $\{d_{XY},d_{YZ},d_{XZ}\}$ gives an admissible metric on $X\sqcup Y\sqcup Z$. Therefore, using the above admissible metrics,

$$d_{GH}(X,Z)\le d_H(X,Z)\le d_H(X,Y)+d_H(Y,Z)\le d_{GH}(X,Y)+d_{GH}(Y,Z)+\epsilon, \qquad (3.10)$$

which completes the proof of the triangle inequality.

Now we are in a position to show that X is isometric to Y if $d_{GH}(X,Y)=0$. The goal is to construct an isometry from X to Y. By $d_{GH}(X,Y)=0$, there exist d^i on $X\sqcup Y$ such that $d^i_H(X,Y)\le 2^{-i}\to 0$. Let $A\equiv\{x_k\}_{k=1}^\infty\subset X$ be countable and dense (Why does A exist?). For $x_1\in A$, let $\{y_{1,i}\}\subset Y$ be a sequence such that $d^i(x_1,y_{1,i})<2^{-i}$. Since (Y,d_Y) is compact, $\{y_{1,i}\}$ has a subsequence $\{y_{1,i_1}\}$ which converges to $y_1\in Y$. Then

$$d_{i_1}(x_1,y_1)\le d_{i_1}(x_1,y_{1,i_1})+d_{i_1}(y_{1,i_1},y_1)\to 0. \qquad (3.11)$$

Similarly, for x_2 and the sequence $\{d_{i_1}\}$, we choose a subsequence $\{d_{i_2}\}\subset\{d_{i_1}\}$ and a point $y_2\in Y$ such that $d_{i_2}(x_2,y_2)\to 0$. Iterating the above process and applying the diagonal argument, one can select a subsequence of admissible metrics $\{d_\ell\}\subset\{d_i\}$ and a sequence of points $y_k\in Y$ such that for any $k\in\mathbb{Z}_+$,

$$d_\ell(x_k,y_k)\to 0,\quad \text{as } \ell\to 0. \qquad (3.12)$$

Now we define $f : A \to Y$ by $f(x_k) \equiv y_k$. Then

$$d_Y(f(x_j), f(x_k)) = d_\ell(f(x_j), f(x_k)) = d_\ell(y_j, y_k) \le d_\ell(y_j, x_j) + d_\ell(x_j, x_k) + d_\ell(x_k, y_k).$$

Taking the limit for $\ell \to \infty$,

$$d_Y(f(x_j), f(x_k)) \le \lim_{\ell\to\infty} d_\ell(x_j, x_k) = \lim_{\ell\to\infty} d_X(x_j, x_k) = d_X(x_j, x_k). \tag{3.13}$$

Using the triangle inequality

$$d_\ell(y_j, y_k) \ge -d_\ell(y_j, x_j) + d_\ell(x_j, x_k) - d_\ell(x_k, y_k), \tag{3.14}$$

one can prove $d_Y(f(x_j), f(x_k)) \ge d(x_j, x_k)$. The above shows that $f : A \to X$ is an isometric embedding. Since A is dense in X, f extends uniquely to an isometric embedding $X \to Y$. One can use the same argument to construct an isometric embedding $h : Y \to X$. □

Theorem 3.5. *We denote by $\mathcal{M}et$ the collection of all isometry classes of compact metric spaces. Then $(\mathcal{M}et, d_{GH})$ is a complete metric space.*

Definition 3.3 (Gromov-Hausdorff convergence)**.** *We say a sequence of compact metric spaces (X_j, d_j) GH-converge to (X, d) if $d_{GH}(X_j, X) \to 0$.*

Definition 3.4 (ϵ-net)**.** *Let (X, d_X) be a metric space. For every $\epsilon > 0$, a subset $X_\epsilon \subset X$ is called ϵ-net of X if X_ϵ is ϵ-dense in X.*

Lemma 3.6. *A complete metric space (X, d_X) is compact iff it is complete and for every $\epsilon > 0$, there exists an ϵ-net X_ϵ with $\#(X_\epsilon) < \infty$.*

Lemma 3.7. *Let $(X_j, d_j) \xrightarrow{GH} (X_\infty, d_\infty)$. Then the following properties hold:*

1. $\operatorname{diam}(X_j) \to \operatorname{diam}(X_\infty) < \infty$.
2. *For any $\epsilon > 0$, there exists some number $N = N(\epsilon) > 0$ such that X_j has an ϵ-net $X_j(\epsilon)$ with $|X_j(\epsilon)| \le N(\epsilon)$ for all j.*

Theorem 3.8 (Gromov's Precompactness Theorem)**.** *A subset $\mathcal{C}$ of $(\mathcal{M}et, d_{GH})$ if compact if and only*

1. *there exists some $D > 0$ such that* $\operatorname{diam}(X) \le D$ *for any $X \in \mathcal{C}$;*
2. *for any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon) > 0$ such that for any $\epsilon > 0$, X has an ϵ-net $X(\epsilon)$ such that $|X(\epsilon)| \le N_0(\epsilon)$.*

Proof. Given any sequence $\{X_j\} \subset \mathcal{C}$, we will select a Cauchy subsequence $\{X_{j_k}\}$ such that for any $\epsilon > 0$, there exists some $N > 0$ such that for all $j_k, j_\ell \ge N$,

$$d_{GH}(X_{j_k}, X_{j_\ell}) < \epsilon. \tag{3.15}$$

Let $X_{j_k}(\epsilon)$, $X_{j_\ell}(\epsilon)$ denote some ϵ-nets, respectively. By triangle inequality,

$$d_{GH}(X_{j_k}, X_{j_\ell}) \le d_{GH}(X_{j_k}(\epsilon), X_{j_\ell}(\epsilon)) + 2\epsilon. \tag{3.16}$$

So it suffices to find a subsequence of $\{X_{j_k}\}$ such that $X_{j_k}(\epsilon)$ converges.

First, let $\epsilon_j \to 0$ be a monotone sequence. We take an ϵ_1-net $X_j(\epsilon_1) \equiv \{x_1^j, \ldots, x_{s_1}^j\}$ of X_j. By assumption, $s_1 \leq N_0(\epsilon)$. Passing to a subsequence, we can just assume $s = s_j$ for all j. Let $d_j \equiv d_{X_j}\Big|_{X_j(\epsilon_1)}$. The matrix $(d_j(x_k^j, x_\ell^j))$ can be viewed as a point in $\mathbb{R}^{s^2}$ and

$$|(d_j(x_k^j, x_\ell^j)) - 0^{s^2}|^2 = \sum_{k,\ell=0}^{s} d_j(x_k^j, x_\ell^j)^2 \leq s \cdot D^2. \tag{3.17}$$

Then there exists a convergent subsequence of the above matrix sequence such that

$$|(d_{j_1}(x_k^{j_1}, x_\ell^{j_1})) - (d_{j_1'}(x_k^{j_1'}, x_\ell^{j_1'}))| < \epsilon_1. \tag{3.18}$$

Therefore, $d_{GH}(X_{j_1}(\epsilon), (X_{j_1'}(\epsilon)) < \epsilon_1$ for all j_1, j_1'.

For $\epsilon_2 > 0$ and $\{X_{j_1}\}$, repeating the above, we have a subsequence $X_{j_2}(\epsilon)$ such that $d_{GH}(X_{j_2}(\epsilon), (X_{j_2'}(\epsilon)) < \epsilon_1$ for all j_2, j_2'. Iterating the above and applying the standard diagonal argument, we can find a subsequence $\{X_\ell\}$ which is a Cauchy sequence.

Finally, for any $\epsilon > 0$, we pick $\epsilon_j < \frac{\epsilon}{3}$. By our construction,

$$d_{GH}(X_k(\epsilon_j), X_\ell(\epsilon_j)) < \epsilon_{\min\{k,\ell\}} < \epsilon_j. \tag{3.19}$$

Then the conclusion follows from the triangle inequality. □

Corollary 3.9 (Gromov)**.** *We denote*

$$\mathcal{M}(n, \kappa, D) \equiv \{(M^n, g) : \operatorname{diam}(M^n) \leq D, \quad \operatorname{Ric}_g \geq (n-1)\kappa\}. \tag{3.20}$$

Then $\mathcal{M}(n, \kappa, D)$ is precompact in $(\mathcal{M}et, d_{GH})$.

Proof. Let $\{p_i\}_{i=1}^N$ be an ϵ-dense subset in M^n such that $B_{\epsilon/5}(p_i) \cap B_{\epsilon/5}(p_j) = \emptyset$ for any $i \neq j$. We take $i_0 \in \{1, \ldots, N\}$ such that

$$\operatorname{Vol}(B_{\frac{\epsilon}{5}}(p_{i_0})) = \min\left\{\operatorname{Vol}(B_{\frac{\epsilon}{5}}(p_i)) : i = 1, \ldots, N\right\}. \tag{3.21}$$

Since the above $\epsilon/5$-balls are disjoint,

$$\operatorname{Vol}(B_D(p_{i_0})) = \operatorname{Vol}(M^n) \geq \sum_{i=1}^{N} \operatorname{Vol}(B_{\epsilon/5}(p_i)) \geq N \cdot \operatorname{Vol}(B_{\epsilon/5}(i_0)). \tag{3.22}$$

Applying Bishop-Gromov's volume comparison, we obtain a uniform bound of N,

$$N \leq \frac{\operatorname{Vol}(B_D(p_{i_0}))}{\operatorname{Vol}(B_{\epsilon/5}(i_0))} \leq \frac{V_\kappa(D)}{V_\kappa(\epsilon/5)} = N(\epsilon, n, D, \kappa). \tag{3.23}$$

Finally, by Theorem 3.8 we conclude the precompactness of $\mathcal{M}(n, \kappa, D)$. □

Definition 3.5 (Pointed Gromov-Hausdorff convergence)**.** *We say a sequence of metric spaces (X_j, d_j, p_j) pointed Gromov-Hausdorff converges to (X, d, p) if*

$$(B_R(p_j), d_j, p_j) \xrightarrow{GH} (B_R(p), d, p), \quad \forall R > 0. \tag{3.24}$$

Example 3.1 (Graph manifold)**.** *(M^3, g_ϵ) satisfies $|\sec_{g_\epsilon}| \leq 1$, $\mathrm{injrad}_{g_\epsilon} \leq \epsilon$, and the diameter is large $\mathrm{diam}_{g_\epsilon}(M^3) \geq \tau(\epsilon)^{-1}$. As $\epsilon \to 0$, there are two types of Gromov-Hausdorff limits: $\mathbb{T}^2 \times \mathbb{R}$ and $\Sigma^2 \times S^1$.*

Graph manifold is an important geometric object in studying the geometry and topology of 3-manifolds. It is a celebrating result that the only collapsing 3-manifolds with curvature uniformly bounded below are graph manifolds. This result, now called *Perel'man's Collapsing Theorem*, was first stated by Perel'man, and first proved by Shioya-Yamaguchi.

Theorem 3.10 (Shioya-Yamaguchi 2001 and 2005)**.** *There exists an absolute constant $\epsilon > 0$ such that if (M^3, g) satisfies $\sec_g \geq -1$ and $\mathrm{Vol}_g(B_1(x)) < v$, then M^3 is diffeomorphic to a graph manifold.*

In completing the proof of *Thurston's Geometrization Conjecture*, Perel'man managed to prove that the Ricci flow gives a *Thick-Thin Decomposition* of M^3 such that the thick part admits a hyperbolic metric and the thin part is volume collapsed and hence diffeomorphic to a graph manifold.

Next, we will give another formulation of Gromov-Hausdorff convergence.

Definition 3.6 (Gromov-Hausdorff approximation)**.** *Let (X, d_X) and (Y, d_Y) be compact metric spaces. A mapping $f : X \to Y$ is called an ϵ-Gromov-Haudorff approximation if the following holds.*

1. *(ϵ-isometric) $|d_Y(f(p), f(q)) - d_X(p, q)| \leq \epsilon$, for all $p, q \in X$;*
2. *(ϵ-onto) $T_\epsilon(f(X)) = Y$.*

We also define

$$\tilde{d}_{GH}(X, Y) \equiv \inf\{\epsilon > 0 : \exists\ \epsilon\text{-}GHAs\ f : X \to Y,\ h : Y \to X\}. \tag{3.25}$$

Lemma 3.11. $\frac{2}{3} \leq d_{GH}(X, Y) \leq \tilde{d}_{GH}(X, Y) \leq 2d_{GH}(X, Y)$.

Theorem 3.12. *Let $(X_j, d_j), (X, d) \in \mathcal{M}et$. Then the following statements are equivalent:*

1. $(X_j, d_j) \xrightarrow{GH} (X, d)$.
2. *There exists a sequence of ϵ_j-GHAs $f_j : (X_j, d_j) \to (X, d)$ such that $\epsilon_j \to 0$.*
3. *There exist ϵ_j-GHAs $f_j : X_j \to X$ and $h_j : X \to X_j$ such that*

$$d(f_j \circ h_j, \mathrm{Id}_X) < \epsilon_j, \quad d_j(h_j \circ f_j, \mathrm{Id}_{X_j}) < \epsilon_j. \tag{3.26}$$

Definition 3.7 (Tangent cone). *Let (X,d) be a metric space and $p\in X$. A metric space (Y,d_*) is called a tangent cone at p if there exists a sequence $\lambda_j\to\infty$ such that*

$$(X,\lambda_j\cdot d,p)\xrightarrow{GH}(Y,d_*) \tag{3.27}$$

Definition 3.8 (Asymptotic cone). *Let (X,d) be a complete non-compact metric space. A metric space (Y,d_*) is called an asymptotic cone (or tangent cone at infinity) if there exists a sequence $\lambda_j\to 0$ such that*

$$(X,\lambda_j\cdot d,p)\xrightarrow{GH}(Y,d_*). \tag{3.28}$$

Example 3.2. *Show that the tangent cone at any point in Riemannian n-manifold is isometric to $\mathbb{R}^n$.*

3.2. **Examples of Gromov-Hausdorff convergence.** This subsection collects some typical examples of Gromov-Hausdorff convergence.

First we introduce some examples of non-collapsing spaces.

Example 3.3 (Asymptotic cone of $\mathbb{Z}\oplus\mathbb{Z}$). *Consider the free abelian group*

$$X\equiv\mathbb{Z}\oplus\mathbb{Z}=\{(m,n):m\in\mathbb{Z},n\in\mathbb{Z}\} \tag{3.29}$$

equipped with the standard discrete metric

$$d\left((m_1,n_1),(m_2,n_2)\right)\equiv|m_1-m_2|+|n_1-n_2|. \tag{3.30}$$

Then

$$(X,j^{-2}d)\xrightarrow{GH}(\mathbb{R}^2,d_{L^1}), \tag{3.31}$$

where

$$d_{L^1}(x,y)\equiv\inf\{\ell(\gamma):\ \gamma \textit{ connects } x,y \textit{ by horizontal and vertical segments}\}. \tag{3.32}$$

Example 3.4 (Asymptotically conic manifold). *Let (M^n,g) satisfy $\mathrm{Ric}_g\geq 0$ and there exists some $v>0$ such that $\mathrm{Vol}_g(B_r(x))\geq vr^n$ for all $r>0$. Then by Cheeger-Colding's theorem, the asymptotic cone of M^n is a metric cone $(C(X),d_C)$ for some compact metric space with* $\mathrm{diam}(X)\leq\pi$.

Example 3.5. *LeBrun-Singer constructed a family of Ricci-flat Kähler metrics g_ϵ on the K3 manifold $\mathcal{K}$ such that $\mathrm{diam}_{g_\epsilon}(\mathcal{K})=1$ and*

$$(\mathcal{K},g_\epsilon)\xrightarrow{GH}(\mathbb{T}^4/\mathbb{Z}_2,d_\infty). \tag{3.33}$$

Example 3.6. *M. Anderson constructed a family of metrics g_ϵ on $M^{2n}\equiv\mathbb{C}P^n\#\mathbb{C}P^n$ with* $\mathrm{Ric}_{g_\epsilon}\geq 2n-1$, $\mathrm{Vol}_{g_\epsilon}(M^{2n})\geq\pi>0$ *and*

$$(X^{2n},g_\epsilon)\xrightarrow{GH}(S^{2n},d_\infty), \tag{3.34}$$

where $\mathrm{diam}(S^{2n})=\pi$ and the limit has two conic singularities.

Next we present some examples of collapsed spaces.

Example 3.7 (Berger sphere)**.**

$$\mathrm{SU}(2) \equiv \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} : z, w \in \mathbb{C},\ |z|^2 + |w|^2 = 1 \right\}. \tag{3.35}$$

Obviously, $\mathrm{SU}(2)$ *is diffeomorphic to* $\mathbb{S}^3$*. We define a map* $\pi : (z, w) \mapsto z/w$*. Then* π *gives a fiber bundle map* $S^1 \to \mathrm{SU}(2) \xrightarrow{\pi} S^2$ *(called Hopf fibration). One can also represents* π *as*

$$(z, w) \mapsto \left(Re(zw), Im(zw), \frac{|z|^2 - |w|^2}{2} \right) \in \mathbb{R}^3. \tag{3.36}$$

There is a natural S^1*-acting on* $\mathrm{SU}(2)$*:* $t \cdot (z, w) \equiv (e^{\sqrt{-1}t} z, e^{-\sqrt{-1}t} w)$*.*

Next we construct a family of collapsing metrics on $\mathrm{SU}(2)$*. Let* X, Y, Z *be the left invariant metrics on* $\mathrm{SU}(2)$ *such that* $[X, Y] = Z$*,* $[Y, Z] = X$*, and* $[Z, X] = Y$*. Indeed, one can just use the Pauli matrices*

$$X = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z = \frac{1}{2} \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}. \tag{3.37}$$

Denote by η_1*,* η_2 *and* η_3 *the dual frames of* X*,* Y*,* Z*, respectively. We define a family of left invariant Riemannian metrics on* $\mathrm{SU}(2)$*,*

$$g_t \equiv t^2 \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3. \tag{3.38}$$

Then we can compute the connection terms,

$$\nabla_X X = 0, \quad \nabla_X Y = (2 - t^2) Z, \quad \nabla_X Z = (-2 + t^2) Y, \tag{3.39}$$

$$\nabla_Y X = -t^2 Z, \quad \nabla_Y Y = 0, \quad \nabla_Y Z = X, \tag{3.40}$$

$$\nabla_Z X = t^2 Y, \quad \nabla_Z Y = -X, \quad \nabla_Z Z = 0. \tag{3.41}$$

Furthermore, $\sec(X, Y) = t^2$*,* $\sec(X, Z) = t^2$*, and* $\sec(Z, Y) = 4 - 3t^2$*.*

Note that, as $t \to 0$*, the direction* X *is collapsing since its integral curve has length* $2t\pi$*. Moreover, the Gromov-Hausdorff limit is the round metric of curvature* $+4$ *on the 2-sphere of radius* $\frac{1}{2}$*.*

Example 3.8 (Klein bottle)**.** *Consider the Klein bottle*

$$K^2 \equiv \{(x, y) \in [0, \epsilon] \times [0, 1]\} / \{(x, 0) \sim (x, 1),\ (0, y) \sim (\epsilon, 1 - y)\} \tag{3.42}$$

equipped with the standard flat metric g_ϵ *such that* $\mathrm{diam}_{g_0}(K^2) = \frac{1}{2}$*. Then*

$$(K^2, g_\epsilon) \xrightarrow{GH} \left[0, \frac{1}{2}\right]. \tag{3.43}$$

Notice that a flat torus cannot collapse to an interval.

Topological computations tell us that

$$\pi_1(K^2) = \langle a, b | abab^{-1} = 1 \rangle \tag{3.44}$$

and $\pi_1(K^2)$ *yields the exact sequence*

$$1 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(K^2) \to \mathbb{Z}_2 \to 1 \tag{3.45}$$

such that $\pi_1(K^2) \cong \mathbb{Z}_2 \ltimes (\mathbb{Z} \oplus \mathbb{Z})$.

4. Structure Theory for Non-Collapsing Einstein Manifolds

4.1. Regularity theory and curvature estimates.

Definition 4.1 (C^1-harmonic radius). *Let (M^n, g) be a Riemannian manifold and let*

$$\Phi = (u_1, \ldots, u_n) : B_r(p) \to B_r(0^n) \subset \mathbb{R}^n \tag{4.1}$$

be a diffeomorphism. For fixed $\epsilon > 0$, Φ is called a C^1-harmonic coordinate system with $\|\Phi\| \le \epsilon$ if the following holds.

(1) *$\Delta_g u_i = 0$ on $B_r(p)$ for any $1 \le i \le n$;*
(2) *let $g_{ij} \equiv g(\nabla u_i, \nabla u_j)$ we have*

$$|g_{ij} - \delta_{ij}|_{C^0(B_r(p))} + r|\nabla g_{ij}|_{C^0(B_r(p))} < \epsilon. \tag{4.2}$$

The C^1-harmonic radius $r_h(p)$ at $p \in M^n$ is defined as

$$r_h(p) \equiv \sup\left\{r > 0 | \exists \text{ a } C^1\text{-harmonic coordinate system } \Phi : B_r(p) \to B_r(0^n)\right\}. \tag{4.3}$$

Lemma 4.1. *In a harmonic coordinate system, we have*

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} + Q(\partial g). \tag{4.4}$$

Lemma 4.2. *Let (M^n, g) satisfy $|\mathrm{Ric}_g| \le \lambda$ and let $\Phi = (u_1, \ldots, u_n) : B_r(p) \to B_r(0^n) \subset \mathbb{R}^n$ be a harmonic coordinate system that satisfies $\|\Phi\| < \epsilon$. Then*

$$r^{1+\alpha}[\nabla g_{ij}]_{C^\alpha(B_{r/2}(p))} \le C(\lambda, \epsilon). \tag{4.5}$$

Lemma 4.3 (Anderson's ϵ-regularity). *Given $n \ge 2$, there exist $\delta = \delta(n) > 0$ and $r_0 = r_0(n) > 0$ such that if (M^n, g) satisfies $|\mathrm{Ric}_g| \le \delta$ and $\mathrm{Vol}_g(B_1(x)) \ge (1-\delta)\mathrm{Vol}_0(B_1)$ for all $x \in B_1(p)$, then for any $q \in B_{1/2}(p)$,*

$$r_h(q) \ge r_0. \tag{4.6}$$

Proof. The lemma is proved by contradiction. Suppose no such a $\delta > 0$ exists. That is, there exist contradiction sequences $\delta_j \to 0$ and (M_j^n, g_j) such that $|\mathrm{Ric}_{g_j}| \le \delta_j$ and for all $x_j \in B_1(p_j)$,

$$\frac{\mathrm{Vol}_{g_j}(B_1(x_j))}{\mathrm{Vol}_0(B_1)} \ge 1 - \delta_j, \tag{4.7}$$

but

$$r_j = r(y_j) \equiv \min\left\{r_h(q_j) : q_j \in B_{1/2}(p_j)\right\} \to 0. \tag{4.8}$$

Let us take the rescaled metric $\tilde{g}_j \equiv r_j^{-2} g_j$. Then

$$r_h^{\tilde{g}_j}(y_j) = 1, \text{ and } r_h^{\tilde{g}_j}(q_j) \ge 1 \text{ for all } q_j \in B_{(2r_j)^{-1}}(p_j)\ . \tag{4.9}$$

Now letting $j \to \infty$, we have

$$(M_j^n, \tilde{g}_j, y_j) \xrightarrow{GH} (X_\infty^n, d_\infty, y_\infty) \tag{4.10}$$

such that $\mathrm{Ric}_\infty \equiv 0$ on X_∞^n and $\mathrm{Vol}_\infty(B_r(x_\infty)) = \mathrm{Vol}_0(B_r)$ for all $r > 0$. Therefore, $X_\infty^n \equiv \mathbb{R}^n$ and hence $r_h(y_\infty) > 1$. Contradiction! □

Theorem 4.4 (Colding's volume convergence)**.** *For every $n \geq 2$, $R > 0$, and $\epsilon > 0$, there exists some $\delta = \delta(n, R, \epsilon) > 0$ such that if (M^n, g) satisfies $\mathrm{Ric}_g \geq -(n-1)$ and $d_{GH}(B_R(p), B_R(0)) < \delta$, then*

$$|\,\mathrm{Vol}_g(B_R(p)) - \mathrm{Vol}_0(B_R(0))| < \epsilon. \tag{4.11}$$

Theorem 4.5 (Cheeger-Colding's ϵ-regularity)**.** *For every $n \geq 2$, there exist numbers $\delta = \delta(n) > 0$ and $r_0 = r_0(n) > 0$ such that if (M^n, g) satisfies $\mathrm{Ric}_g \geq -(n-1)$ and $d_{GH}(B_2(p), B_2(0)) < \delta$, then*

$$r_h(q) \geq r_0 \quad \forall q \in B_1(p). \tag{4.12}$$

Theorem 4.6 (Anderson 1992)**.** *Given $n \geq 2$, there exist $\epsilon = \epsilon(n) > 0$ and $C(n) > 0$ such that if an Einstein manifold (M^n, g) satisfies $\mathrm{Ric}_g \geq -(n-1)g$, and*

$$\frac{\mathrm{Vol}_0(B_{2r})}{\mathrm{Vol}_g(B_{2r}(p))} \int_{B_{2r}(p)} |\,\mathrm{Rm}_g\,|^{\frac{n}{2}} < \epsilon, \tag{4.13}$$

then

$$\sup_{B_r(p)} r^2 |\,\mathrm{Rm}_g\,| \leq C(n) \left(\frac{\mathrm{Vol}_0(B_{2r})}{\mathrm{Vol}_g(B_{2r}(p))} \int_{B_{2r}(p)} |\,\mathrm{Rm}_g\,|^{\frac{n}{2}} \right)^{\frac{2}{n}}. \tag{4.14}$$

Remark 4.1. *It follows from theorem 4.4 of Anderson's paper: The L^2-structure of ...*

4.2. **Metric cone structure and singular set of non-collapsed Ricci limits.**

Definition 4.2 (Metric cone (Euclidean cone))**.** *Let (Σ, d_Σ) be a compact metric space with diameter $\leq \pi$. The metric space $(Z, d_C) \equiv (C(\Sigma), d_C)$ is called the metric cone over the cross-section Σ, denoted by $C(\Sigma) \equiv C_0(\Sigma)$, if Z is homeomorphic to the topological cone $(\Sigma \times [0,\infty))/(\Sigma \times \{0\})$ and the cone metric d_C is given by*

$$d^2(x,y) = d^2(z_*, x) + d^2(z_*, y) - 2d(z_*, x)d(z_*, y)\cos d_\Sigma(\bar{x}, \bar{y}), \tag{4.15}$$

where $z_* \equiv \Sigma \times \{0\}$ is called the cone vertex of $C(\Sigma)$.

Lemma 4.7. *Any cone metric g_C is scaling invariant.*

Example 4.1. *Let (Σ, h) be a compact Riemannian manifold with $\mathrm{diam}_h(\Sigma) \leq \pi$. The cone metric of $(C(\Sigma), d_C, z_*)$ can be written by the warped Riemannian metric $g_C = dr^2 + r^2 \cdot h$ away from the cone tip z_*. As an exercise, one can prove the following Euclidean law of cosine on $(C(\Sigma), g_C)$.*

Example 4.2. *Let $g_C = dr^2 + r^2 \cdot h$ be the cone metric of $C(\Sigma)$. Show that $\mathfrak{L}_{\partial_r} g = \frac{2}{r} g$.*

Example 4.3. *Prove that a metric cone $C(\Sigma)$ is smooth everywhere if and only if $C(\Sigma)$ is flat which is equivalent to say the cross-section Σ is isometric to the round sphere of curvature $+1$.*

There are many examples of metric cones:

1. $\mathbb{R}^n$ is the metric cone over the standard round sphere $\mathbb{S}^{n-1}$.
2. $\mathbb{R}_+$ is the metric cone over a point.
3. The half plane $\mathbb{R} \times \mathbb{R}_+$ is the metric cone over the segment $[0, \pi]$. It can be viewed as the quotient $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the reflection $(x, y) \mapsto (x, -y)$.
4. Let $\mathbb{Z}_+$ be the group generated by the rotation $(x, y) \mapsto (-x, -y)$. Then $\mathbb{R}_+/\mathbb{Z}_2$ is the metric cone over the circle S^1 of perimeter π.
5. The metric cone $\mathbb{R}^4/\mathbb{Z}_2 \cong C(\mathbb{R}P^3)$, where the quotient group $\mathbb{Z}_2$ is generated by the involution $\iota : (x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4)$.

Using the curvature equations, we can compute the curvature of the cross-section of $C(\Sigma)$.

Example 4.4. *Let $(C(\Sigma), g_C, z_*)$ be a metric cone over a compact manifold (Σ, h), where z_* is the cone tip. Prove that away from the cone tip z_*, $\mathrm{Ric}_{g_C} \equiv 0$ iff $\mathrm{Ric}_h \equiv (n-2)h$, and g_C is flat iff $\sec_h \equiv +1$.*

Definition 4.3 (Spherical suspension)**.** *Let (Σ, d_Σ) be a metric space of diameter $\le \pi$. A metric space (Z, d) is called the spherical suspension (or spherical cone) over Σ, denoted by $\mathrm{Susp}_{+1}(\Sigma) \equiv C_{+1}(\Sigma)$, if Z is homeomorphic to $(\Sigma \times [0, \pi])/(\Sigma \times \{0, \pi\})$ and the spherical metric d is given by*

$$\cos d(x, y) = \cos d(z_*, x) \cos d(z_*, y) + \sin d(z_*, x) \sin d(z_*, y) \cos d_\Sigma(\bar{x}, \bar{y}). \tag{4.16}$$

Notice that, there are two vertices $z_* \equiv \Sigma \times \{0\}$ and $w_* \equiv \Sigma \times \{\pi\}$ on the spherical suspension $C_{+1}(\Sigma)$.

Theorem 4.8. *Let $C(Z)$ be a metric cone over a compact Riemannian manifold (Z, h). Assume that $C(Z)$ is isometric to $C(W) \times \mathbb{R}$. Then both Z and W are round spheres of curvature $+1$. Moreover, Z is the spherical suspension over W.*

Definition 4.4 (Hyperbolic suspension)**.** *Let (Σ, d_Σ) be a metric space of diameter $\le \pi$. A metric space (Z, d) is a called the hyperbolic suspension (or hyperbolic cone) over Σ, denoted by $\mathrm{Susp}_{-1}(\Sigma) \equiv C_{-1}(\Sigma)$ if Z is homeomorphic to the topological cone $(\Sigma \times [0, \infty))/(\Sigma \times \{0\})$ and the is given by*

$$\cosh d(x, y) = \cosh d(z_*, x) \cosh d(z_*, y) - \sinh d(z_*, x) \sinh d(z_*, y) \cos d_\Sigma(\bar{x}, \bar{y}), \tag{4.17}$$

where $z_* \equiv \Sigma \times \{0\}$ is called the cone vertex of $C_{-1}(\Sigma)$.

Example 4.5. *Let* $Z \equiv \mathrm{Susp}_k(\Sigma)$ *with* $k \in \{-1, 1\}$. *Show that* $\sec_\Sigma \equiv 1$ *if and only if* $\sec_Z \equiv k$.

Theorem 4.9 (Almost Volume Cone Implies Metric Cone, Cheeger-Colding 1996)**.** *Given* $n \geq 2$ *and* $\epsilon > 0$, *there exists* $\delta = \delta(n, \epsilon) > 0$ *such that the following holds. Let* (M^n, g) *satisfy* $\mathrm{Ric} \geq -(n-1)\delta^2$ *such that for some* $p \in M^n$

$$|Q_{2r}(p) - Q_r(p)| < \delta. \tag{4.18}$$

Then there exists a compact length metric space (X, d_X) *with* $\mathrm{diam}(X) \leq \pi$ *such that*

$$d_{GH}(B_r(p), B_r(x_*)) < \epsilon, \quad B_r(x_*) \subset C(X). \tag{4.19}$$

Corollary 4.10. *Let* (X^n_∞, d_∞) *be a non-collapsed Ricci-limit space. Then all tangent cones on* X^n_∞ *are metric cones.*

Corollary 4.11. *For any* $n \geq 2$ *and* $\epsilon > 0$, *there exists* $\delta(n, \epsilon) > 0$ *such that if* (M^n, g, p) *satisfies* $\mathrm{Ric}_g \geq -(n-1)$ *and*

$$|\mathrm{Vol}_g(B_2(p)) - \mathrm{Vol}_0(B_2(0^n))| < \delta, \tag{4.20}$$

then

$$d_{GH}(B_1(p), B_1(0^n)) < \epsilon. \tag{4.21}$$

Corollary 4.12 (Uniqueness)**.** *Let* (X, d) *be a non-collapsed Ricci-limit space. If a point* $p \in X$ *has a tangent cone* $\mathbb{R}^n$, *then every tangent cone at* x *is isometric to* $\mathbb{R}^n$.

Theorem 4.13 (Cheeger-Colding 1997)**.** *Let* (M^n_j, g_j, p_j) *be a sequence of Riemannian manifolds with* $\mathrm{Ric}_{g_j} \geq -(n-1)g_j$ *and* $\mathrm{Vol}_{g_j}(B_1(p_j)) \geq v > 0$. *Then there exists a length metric space* $(X_\infty, d_\infty, p_\infty)$ *such that the following holds.*

(1) $\dim_{\mathcal{H}}(X_\infty) = n$.
(2) *Passing to a subsequence,*

$$(M^n_j, g_j, p_j) \xrightarrow{GH} (X_\infty, d_\infty, p_\infty) \text{ and } \mathrm{Vol}_{g_j}(B_R(p_j)) \to \mathcal{H}^n(B_R(p_\infty)). \tag{4.22}$$

(3) *The singular set* $\mathcal{S}$ *of* X^n_∞ *admits the stratification* $\mathcal{S} = \mathcal{S}^{n-2}$ *and* $\dim_{\mathcal{H}}(\mathcal{S}^k) \leq k$.

Definition 4.5. *Let* (X, d) *be a Ricci-limit space. We define*

$$\mathcal{R} \equiv \{x \in X : \text{ every tangent cone at } x \text{ is isometric to } \mathbb{R}^n\}. \tag{4.23}$$

The set $\mathcal{S} \equiv X \setminus \mathcal{R}$ *is called the singular set of* X.

Definition 4.6 (Classical stratification)**.** *Let* (X, d) *be a Ricci-limit metric space. We define*

$$\mathcal{S}^k \equiv \left\{x \in X : \text{ no tangent cone at } x \text{ isometrically splits off } \mathbb{R}^{k+1}\right\}. \tag{4.24}$$

Lemma 4.14. *Let (X^n, d) be a non-collapsed Ricci-limit space. Then it holds that*

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \ldots \subset \mathcal{S}^{n-1} = \mathcal{S}. \tag{4.25}$$

That is, $\mathcal{S} \setminus \mathcal{S}^{n-1} = \emptyset$.

Theorem 4.15 (Cheeger-Colding 1997)**.** *Let $(M_j^n, g_j, p_j) \xrightarrow{GH} (X_\infty^n, d_\infty, p_\infty)$ be a non-collapsing sequence with $|\operatorname{Ric}_{g_j}| \leq n-1$. Then $X_\infty^n \setminus \mathcal{S}$ is a smooth manifold equipped with a $C^{1,\alpha}$-Riemannian metric.*

Theorem 4.16 (Cheeger 2003)**.** *Let (M_j^n, g_j, p_j) be a sequence satisfying $\operatorname{Ric}_{g_j} \geq -(n-1)$, $\operatorname{Vol}_{g_j}(B_1(p_j)) \geq v > 0$, and*

$$\int_{B_2(p_j)} |\operatorname{Rm}_{g_j}|^p \leq \Lambda, \tag{4.26}$$

such that $(M_j^n, g_j) \xrightarrow{GH} (X_\infty^n, d_\infty)$. Then $\dim_{\mathcal{H}}(\mathcal{S}) \leq n - 2p$. In particular, in the Kähler case, $\dim_{\mathcal{H}}(\mathcal{S}) \leq n-4$.

Theorem 4.17 (Codimension-4 Regularity, [?])**.** *Let $(M_j^n, g_j, p_j) \xrightarrow{GH} (X_\infty^n, d_\infty, p_\infty)$ be a non-collapsing sequence with $|\operatorname{Ric}_{g_j}| \leq n-1$. Then $\mathcal{S} = \mathcal{S}^{n-4}$.*

Corollary 4.18. *Let (X_∞^n, d_∞) be a non-collapsed Einstein limit. Then there exists a closed subset $\mathcal{S} \subset X_\infty^n$ such that $X_\infty^n \setminus \mathcal{S}$ admits a smooth Einstein metric g_∞ with $d_{g_\infty} = d_\infty$.*

Theorem 4.19 (Cheeger-Naber's Diffeomorphism Finiteness Theorem, 2015)**.** *Given $v > 0$, $D > 0$, and $\kappa > 0$, there exists $N(v, D, \kappa) > 0$ such that the class*

$$\mathcal{M}_{\operatorname{Ric}}(4, v, D, \kappa) \equiv \left\{(M^4, g) : |\operatorname{Ric}_g| \leq \kappa,\ \operatorname{diam}_g(M^4) \leq D,\ \operatorname{Vol}_g(M^4) \geq v > 0\right\} \tag{4.27}$$

has at most N diffeomorphism types. As a corollary,

$$\int_{M^4} |\operatorname{Rm}_g|^2 \leq C(v, D, \kappa) \tag{4.28}$$

for any manifold $(M^4, g) \in \mathcal{M}_{\operatorname{Ric}}(4, v, D, \kappa)$.

Definition 4.7 (Quantitative stratification)**.** *Given $0 \leq k \leq n-2$ and $\epsilon > 0$, we define*

$$\mathcal{S}_\epsilon^k \equiv \bigcap_{r>0} \mathcal{S}_{\epsilon,r}^k \equiv \{x \in B_1(p) : \textit{for no } r \in (0,1) \textit{ is } B_r(x) \textit{ a } (k+1, \epsilon)\textit{-symmetric ball}\}. \tag{4.29}$$

Theorem 4.20 (Jiang-Naber 2018)**.** *For any $n \geq 2$ and $v > 0$, there exists $C(n, v) > 0$ such that the following holds. If (M^n, g, p) satisfies $|\operatorname{Ric}_g| \leq n-1$, $\operatorname{Vol}_g(B_1(p)) \geq v > 0$, then*

$$\fint_{B_1(p)} |\operatorname{Rm}_g|^2 \leq C. \tag{4.30}$$

Theorem 4.21 (Cheeger-Jiang-Naber 2021)**.** *Let* $(M^n_j, g_j, p_j) \xrightarrow{GH} (X^n_\infty, d_\infty, p_\infty)$ *satisfy*

$$\mathrm{Vol}_{g_j}(B_1(p_j)) \geq v > 0, \quad \mathrm{Ric}_{g_j} \geq -(n-1)g_j. \tag{4.31}$$

Then $\mathcal{S}^k$ *is* k*-rectifiable and for* $\mathcal{H}^k$*-a.e.* $x \in \mathcal{S}^k$*, every tangent cone at* x *is* k*-symmetric.*

Theorem 4.22 (ϵ-Stratification and manifold structure)**.** *Let* $(M^n_j, g_j, p_j) \xrightarrow{GH} (X^n_\infty, d_\infty, p_\infty)$ *satisfy* $\mathrm{Vol}_g(B_1(p_j)) \geq v > 0$ *and* $\mathrm{Ric}_{g_j} \geq -(n-1)$*. Then for any* $\epsilon > 0$*, there exists* $C_\epsilon = C_\epsilon(n, v, \epsilon) > 0$ *such that*

$$\mathrm{Vol}(T_r(\mathcal{S}^k_\epsilon(X)) \cap B_1(p_\infty)) < C_\epsilon \cdot r^{n-k}. \tag{4.32}$$

In particular, $\mathcal{H}^k(\mathcal{S}^k_\epsilon \cap B_1(p_\infty)) \leq C_\epsilon$*. Moreover, the following holds:*

1. *the set* $\mathcal{S}^k_\epsilon$ *is* k*-rectifiable, and for* $\mathcal{H}^k$*-a.e.* $x \in \mathcal{S}^k_\epsilon$ *every tangent cone at* x *is* k*-symmetric.*
2. $X^n \setminus \mathcal{S}_\epsilon$ *is bi-Hölder homeomorphic to a smooth manifold, where* $\mathcal{S}_\epsilon \equiv \bigcup_{k=0}^{n-2} \mathcal{S}^k_\epsilon$*.*

5. Collapsing Manifolds with Bounded Curvature

Fundamental structures in the geometry of collapsing manifolds with bounded curvature include the nilpotent fibration structure and Cheeger-Fukaya-Gromov's nilpotent Killing structure. In this section, we will apply Cheeger-Colding's harmonic splitting map to construct a fibration whose collapsing fibers are almost flat manifolds.

5.1. An example.

Let us first discuss an elementary example of product manifold.

Theorem 5.1. *Let* (M^n, g) *be a complete Riemannian manifold. Then there exists a full measure subset* $\mathscr{R}$ *in* $M^n \times M^n$ *with respect to the product measure such that for any* $(p, q) \in \mathscr{R}$*, there exists a unique minimal geodesic connecting* p *and* q*.*

The following lemma is also standard.

Lemma 5.2. *Let* $p = (x_1, y_1)$ *and* $q = (x_2, y_2)$ *be any points on the product manifold* $(M^n \times N^k, g \oplus h)$*. Then*

$$d^2_{g\oplus h}(p, q) = d^2_g(x_1, x_2) + d^2_h(y_1, y_2). \tag{5.1}$$

Example 5.1. *Now we consider a special case of Riemannian product. Let* (X^n, g) *be a Riemannian manifold. We define the Riemannian product* $Y^{n+1} \equiv \mathbb{R} \times X^n$ *with a natural coordinate system* (t, x)*. We define a function* $h(t, x) \equiv t$*. Then one can check that*

$$|\nabla h| \equiv 1, \quad |\nabla^2 h| \equiv 0. \tag{5.2}$$

We take three points $z, x, w \in Y^{n+1}$ *such that* $h(x) = 0$ *and* w *is the orthogonal projection of* z *onto the fiber* $X^n \times \{0\}$*. In particular,* $h(w) = 0$ *as well. Let* $d_0 \equiv d(z, w)$*. We will prove* *that $d^2(z,x) = d^2(z,w) + d^2(x,w)$. NOTE: we will always work with the "regular points" as described in Theorem 5.1.*

Let $\sigma : [0, d_0] \to Y^{n+1}$ be the unique minimal geodesic connecting w and z. For any $s \in [0, d_0]$, there exists a geodesic $\tau_s : [0, \ell(s)] \to Y^{n+1}$ with $\tau_s(0) = x$ and $\tau_s(\ell(s)) = \sigma(s)$. Then we have that

$$\begin{aligned} d^2(z,w) = d_0^2 &= 2\int_0^{d_0} s ds \\ &= 2\int_I (h(\sigma(s)) - h(\sigma(0)))ds \\ &= 2\int_I (h(\tau_s(\ell_s)) - h(\tau_s(0)))ds \\ &= 2\int_I \int_0^{\ell_s} \langle \nabla h(\tau_s(t)), \tau_s'(t)\rangle dt ds. \end{aligned} \tag{5.3}$$

Next, we show that $\langle \nabla h, \tau_s' \rangle$ is constant along the geodesic τ_s. In fact, for any $t \in [0, s]$

$$\Big|\langle \nabla h(\tau_s(t)), \tau_s'(t)\rangle - \langle \nabla h, \tau_s'(t)\rangle(\tau_s(\ell_s))\Big| = \Big|\int_t^s \langle \nabla_{\tau_s'(u)} \nabla h, \tau_s'(u)\rangle du\Big| \equiv 0. \tag{5.4}$$

Therefore,

$$d^2(z,w) = 2\int_I \int_0^{\ell_s} \langle \nabla h, \tau_s'\rangle(\sigma_s) dt ds = 2\int_I \ell_s \langle \sigma', \tau_s'\rangle(\sigma(s)) ds. \tag{5.5}$$

Applying the first variation formula, $\frac{d}{ds}\ell_s = \langle \sigma', \tau_s'\rangle$, which implies that

$$d^2(z,w) = 2\int_I \ell_s \ell_s' ds = 2\int_0^{d_0} \ell_s \ell_s' ds = \ell^2(d_0) - \ell^2(0) = d^2(x,z) - d^2(x,w). \tag{5.6}$$

5.2. **Isometric splitting and quantitative splitting.**

Definition 5.1 (Killing field)**.** *Let (M^n, g) be a Riemannian manifold. A vector field X is called a Killing field if and only if its flow acts on M^n by isometries.*

Lemma 5.3. *A vector field X on (M^n, g) is Killing iff $\mathfrak{L}_X g \equiv 0$.*

Lemma 5.4 (Splitting Lemma)**.** *Let (M^n, g) be a complete Riemannian manifold. (M^n, g) is isometric to the Riemann product if and only if there exists a function $f \in C^2(M^n)$ that satisfies $|\nabla f| \equiv 1$ and $|\nabla^2 f| \equiv 0$.*

Proof. We only prove the "only if" part. First, M^n is non-compact since otherwise $\nabla f = 0$ when f attains the maximum. Next, we will show that ∇f is a Killing field. To see this, taking any vector fields $Y, Z \in \mathfrak{X}(M^n)$,

$$\begin{aligned}(\mathfrak{L}_{\nabla f} g)(Y,Z) &= \nabla f(g(Y,Z)) - g([\nabla f, Y], Z) - g(Y, [\nabla f, Z]) \\ &= g(\nabla_Y \nabla f, Z) + g(Y, \nabla_Z \nabla f) \\ &= 2\nabla^2 f(Y,Z) \equiv 0. \end{aligned} \tag{5.7}$$

Notice that $df(\nabla f) = |\nabla f|^2 \equiv 1$, which implies $f : M^n \to \mathbb{R}$ is submersion. It follows that $N^{n-1} \equiv f^{-1}(0)$ is a (totally geodesic) submanifold.

Now we define a map $\Phi : N \times \mathbb{R} \to M^n$ by $\Phi(x,t) \equiv \gamma_x(t)$, where γ_x is the integral curve of ∇f at x. Since ∇f is a Killing field, $\Phi(x,t)$ is an isometry. □

Definition 5.2 (Upper barrier function)**.** *Let (M^n, g) be a Riemannian manifold, let U be an open neighborhood of $p \in M^n$. For any $f \in C^0(U)$, a function $g_p \in C^2(U)$ is said to be an upper barrier function of f at p if*

$$g_p(p) = f(p), \quad f(x) \le g_p(x), \quad \forall x \in U. \tag{5.8}$$

Definition 5.3 (Weak derivative)**.** *Let $f : (a,b) \to \mathbb{R}$ be continuous. We say $f''(t_0) \le \lambda$ in the sense of barrier function for some $t_0 \in (a,b)$ if for any $\epsilon > 0$, there exists some barrier function g_ϵ of f at t_0 such that $g_\epsilon''(t_0) \le \lambda + \epsilon$.*

Definition 5.4. *Let (M^n, g) be a Riemannian manifold and $f \in C^0(M^n)$ and let $p \in M^n$. If $(f \circ \gamma)'' \le \lambda$ holds in the barrier sense for any geodesic γ starting from p, then we say $\nabla^2 f \le \lambda \cdot \mathrm{Id}$ holds in the barrier sense on M^n.*

Theorem 5.5 (Calabi-Hopf's strong maximum principle)**.** *Let $f \in C^0(M^n)$ satisfy $\Delta f \le 0$ in the barrier sense. Then f is local constant at a local minimum, and f is a constant if f has a global minimum.*

Theorem 5.6 (Cheeger-Gromoll)**.** *Let (M^n, g) be a complete non-compact Riemannian manifold with $\mathrm{Ric}_g \ge 0$. If (M^n, g) admits a line, then M^n is isometric to the Riemann product $\mathbb{R}^k \times N$ for some $k \in \mathbb{Z}_+$, where N is a Riemannian with $\mathrm{Ric}_N \ge 0$ and contains no line.*

To prove the theorem, we need a few preliminary results.

We define the Busemann function as follows. Let $\gamma : [0, \infty) \to M^n$ be a ray with $\gamma(0) = p$. For any $t \ge 0$, we define

$$b_\gamma^t(x) \equiv t - d(x, \gamma(t)), \quad x \in M^n. \tag{5.9}$$

Then the following properties hold.

(1) $b_\gamma^t(x) \le d(p,x)$.
(2) b_γ^t is 1-Lipschitz, i.e.,

$$|b_\gamma^t(x) - b_\gamma^t(y)| \le d_g(x,y). \tag{5.10}$$

(3) $b^t_\gamma(x)$ is increasing in t.

By Arzelà-Ascoli, the limiting function $b_\gamma \equiv \lim_{t\to+\infty} b^t_\gamma$, called the *Busemann function determined by* γ, is well-defined and b_γ is also 1-Lipschitz.

Lemma 5.7. *Let* (M^n, g) *be a complete non-compact Riemannian manifold with* $\mathrm{Ric}_g \geq 0$. *Let* $\gamma : [0, \infty) \to M^n$ *be a ray. Then the Busemann function* b_γ *satisfies* $\Delta b_\gamma \geq 0$ *in the barrier sense.*

Proof. First, we describe the construction of the barrier function. Given $q \in M^n$, we pick a sequence $t_i \to +\infty$ and we pick a sequence of minimizing geodesic σ_i connecting q and $\gamma(t_i)$. By the compactness of $\mathbb{S}^{n-1}$, $\{\sigma_i'(0)\}$ has a converging subsequence which converges to a limiting vector v. Then $\sigma(t) \equiv \mathrm{Exp}_q(tv)$ gives a ray starting from $q \in M^n$.

Let us choose a function

$$b^t_{\gamma,q}(x) \equiv b_\gamma(q) + t - d_g(x, \sigma(t)). \tag{5.11}$$

First, we prove the following properties.

(1) $b^t_{\gamma,q}$ is a lower barrier function of b_γ at q.
(2) $b^t_{\gamma,q}$ is smooth at q when $t > 0$.

Since b_γ is 1-Lipschitz,

$$b^t_{\gamma,q}(x) = b_\gamma(q) + t - d_g(x, \sigma(t)) = b_\gamma(\sigma(t)) - d_g(x, \sigma(t)) \leq b_\gamma(x). \tag{5.12}$$

This completes the proof of item (1). To show item (2), observe that σ is a ray, and hence q is not in the cut locus of $\sigma(t)$. Then the distance function $d_{\sigma(t)}$ is smooth around q, which completes the proof of item (2).

Next, by Laplacian comparison, for any $t > 0$,

$$\Delta b^t_{\gamma,q}(q) = -\Delta d(\sigma(t), q) \geq -\frac{n-1}{d(\sigma(t), q)} = -\frac{n-1}{t}. \tag{5.13}$$

Therefore, $\Delta b_\gamma(q) \geq 0$ holds in the barrier sense. □

Proof of Theorem 5.6. We write the line $\gamma \subset M^n$ with $\gamma(0) = p$ as the union of the two rays $\gamma \equiv \gamma_- \cup \gamma_+$. Let us denote $b_+ \equiv b_{\gamma_+}$ and $b_- \equiv b_{\gamma_-}$. The main point is to show that b_+ and b_- satisfy

$$|\nabla b_\pm| \equiv 1, \quad |\nabla^2 b_\pm| \equiv 0. \tag{5.14}$$

First, by definition, $(b_+ + b_-)(\gamma(s)) = 0$ for any $s \in (-\infty, \infty)$. Moreover, by triangle inequality, one can check that $(b_+ + b_-)(x) \leq 0$ for any $x \in M^n$. Applying Lemma 5.7,

$$\Delta(b_- + b_+) \geq 0 \tag{5.15}$$

holds in the barrier sense. It follows from the Calabi-Hopf maximum principle (Theorem 5.5) that $(b_+ + b_-)(x) = 0$ for any $x \in M^n$. Therefore, both b_+ and b_- are harmonic.

Immediately, b_+ and b_- are both smooth on M^n. It follows that $|\nabla b_\pm| \equiv 1$. Next, applying Bochner's formula

$$0 = \frac{1}{2}\Delta|\nabla b_\pm|^2 = |\nabla^2 b_\pm|^2 + \mathrm{Ric}(\nabla b_\pm, \nabla b_\pm). \tag{5.16}$$

Combining with the curvature assumption $\mathrm{Ric}_g \geq 0$, we conclude that $|\nabla^2 b_\pm| \equiv 0$. □

Theorem 5.8. *Let (M^n, g) be a closed Riemannian manifold with $\mathrm{Ric}_g \geq 0$. Then up to a finite cover, M^n is diffeomorphic to $\mathbb{T}^k \times N^{n-k}$. In particular, $\pi_1(M^n)$ is virtually free abelian.*

5.3. **Fibration theorems.** In the theory of collapsing manifolds with bounded curvature, the first first milestone theorem is Gromov's Almost Flat Manifolds Theorem.

Theorem 5.9 (Gromov 1978)**.** *For any $n \geq 2$, there exist constants $\epsilon = \epsilon(n) > 0$ and $w = w(n)$ such that if (M^n, g) satisfies*

$$\mathrm{diam}_g(M_j^n)^2 \cdot \max_{M^n} |\sec_{g_j}| \leq \epsilon, \tag{5.17}$$

then there exists a finite normal covering space $\widehat{M}^n$, of index bounded by $w(n)$, which is diffeomorphic to a nilmanifold.

In 1987, Fukaya proved a fibration theorem for collapsing manifolds with bounded curvatrue, where the limit spaces are assumed to be smooth.

Theorem 5.10 (Fukaya 1987)**.** *Let (M_j^n, g_j) satisfy $|\sec_{g_j}| \leq 1$ and $(M_j^n, g_j) \xrightarrow{GH} (Y_\infty^k, g_\infty)$, where the limit is a smooth Riemannian manifold. Then there exists a large $J_0 \in \mathbb{Z}_+$ such that the following holds for any $j \geq J_0$:*

1. *there exists a fiber bundle $N_j^{n-k} \to M_j^n \xrightarrow{F_j} Y^k$;*
2. *if $d_{GH}(M_j^n, Y_\infty^k) \leq \epsilon_j \to 0$, then $\mathrm{diam}(F_j) \leq \tau(\epsilon_j)$ with $\lim_{j\to\infty} \tau(\epsilon_j) = 0$;*
3. *there exists a uniform constant $C_0 > 0$ such that $|\mathrm{II}_{N_j^{n-k}}| \leq C_0$.*
4. *F_j is an almost Riemannian submersion, in the sense that for any vector v orthogonal to the fiber of F_j, we have*

$$(1 - \tau_j)|v|_{g_j} \leq |dF_j(v)|_{g_\infty} \leq (1 + \tau_j)|v|_{g_j}. \tag{5.18}$$

In particular, N_j^{n-k} is almost flat and hence diffeomorphic to an infranilmanifold.

Two years later, Fukaya was able to treat general limit spaces and managed to construct a singular fibration for collapsing manifolds with bounded curvature.

Theorem 5.11 (Fukaya 1989)**.** *Let $(M_j^n, g_j) \xrightarrow{GH} (X_\infty^k, d_\infty)$ satisfy $k < n$ and $|\sec_{g_j}| \leq 1$. Then one has the following equivariant convergence diagram:*

$$\begin{tikzcd}
\widehat{N} \arrow[r] \arrow[d] & (F(M_j), O(n)) \arrow[r, "eqGH"] \arrow[d, "\pi_j"] & (Y_\infty, O(n)) \arrow[d, "\pi_\infty"] \\
N \arrow[r] & (M_j^n, g_j) \arrow[r, "GH"] & (X_\infty, d_\infty)
\end{tikzcd}$$

and there exists a singular fibration $N \to M_j \to X_\infty$ with N as its generic fiber, where Y_∞ is smooth manifold equipped with a $C^{1,\alpha}$-Riemannian metric, $\widehat{N}$ is a nilmanifold, and N is finitely covered by a nilmanifold.

In the following, we will use Cheeger-Colding's harmonic splitting map to prove Theorem 5.10.

Theorem 5.12 (Cheeger-Colding's harmonic splitting map)**.** *Given any $\epsilon > 0$ and $n \geq 2$, there exists some $\delta = \delta(n, \epsilon) > 0$ such that the following holds. If (M^n, g, p) is a Riemannian manifold satisfying $\mathrm{Ric}_g \geq -(n-1)\delta$ and $d_{GH}(B_4(p), B_4(0^d)) < \delta$, $B_4(0^4) \subset \mathbb{R}^d$, then there exists a harmonic map*

$$\Phi = (u^{(1)}, \ldots, u^{(d)}) : B_2(p) \to \mathbb{R}^d$$

such that the following properties hold:

1. $\Phi : B_2(p) \to B_2(0^d) \subset \mathbb{R}^d$ *is an ϵ-Gromov-Hausdorff approximation;*
2. $|\nabla u^{(\alpha)}|(x) \leq 1 + \epsilon$ *holds for any $x \in B_2(p)$ and $1 \leq \alpha \leq d$;*
3. *The following estimate holds*

$$\sum_{\alpha,\beta=1}^{d} \fint_{B_2(p)} |\langle \nabla u^{(\alpha)}, \nabla u^{(\beta)} \rangle - \delta_{\alpha\beta}| \, \mathrm{dvol}_g + \sum_{\alpha=1}^{d} \fint_{B_2(p)} |\nabla^2 u^{(\alpha)}|^2 \, \mathrm{dvol}_g < \epsilon.$$

We will also need a good cutoff function with uniform derivative estimates. Here we briefly review the standard heat flow regularization, and we refer to Lemma 3.1 of [**?**] for results on general RCD spaces.

Lemma 5.13. *Let (X^n, g) be a Riemannian manifold with $\mathrm{Ric}_g \geq 0$. Assume that for any $m \in \mathbb{N}$, there exists a constant $\Lambda_m > 0$ such that $|\nabla^m \mathrm{Rm}_g| \leq \Lambda_m$ uniformly on X. Then for any $p \in X$ and $r \in (0, 1]$ with $\overline{B_{2r}(p)}$ compact, there exists a cut-off function $\psi : X^n \to [0, 1]$ which satisfies the following properties*

1. $\psi \equiv 1$ *on $B_r(p)$ and $\psi \equiv 0$ on $X \setminus B_{2r}(p)$.*
2. *For any $m \in \mathbb{Z}_+$, there exists a constant $C = C(m, n) > 0$ such that $r^m |\nabla^m \psi| \leq C$.*

Proof. Without loss of generality suppose $r=1$. The proof below can be made purely local, but to simplify notations we assume X is complete. For any $q\in X$, we first take a cutoff function ρ defined by

$$\rho(y)=\begin{cases}1, & y\in B_1^g(q),\\ 2-d_{g_j}(y,q), & y\in A_{1,2}^g(q),\\ 0, & y\in X\setminus B_2^g(q).\end{cases}$$

For $t>0$, consider the heat flow $\psi_t\equiv H_t(\rho)$ of the 1-Lipschitz cutoff function ρ. It is standard that on X we have the pointwise estimate $|\nabla_g\psi_t|^2+\frac{2t}{n}(\Delta_g\psi_t)^2\leq 1$. Then for all $y\in X$, we have

$$|\psi_t(y)-\rho(y)|\leq\int_0^t|\Delta_g\psi_s(y)|ds\leq\sqrt{2nt}.$$

Now fix $\tau=\frac{1}{18n^2}$. Then $\psi_\tau(y)\in[\frac{2}{3},1]$ for $y\in B_1(q)$ and $\psi_\tau(y)\in[0,\frac{1}{3}]$ for $y\in X\setminus B_2(q)$.

Next, we choose a smooth cutoff function $h:[0,1]\to[0,1]$ which satisfies

$$h(s)=\begin{cases}1, & \frac{2}{3}\leq s\leq 1,\\ 0, & 0\leq s\leq\frac{1}{3},\end{cases}$$

and set $\psi=h\circ\psi_\tau$. Since ψ_t solves the heat equation, the higher order derivative estimate of ψ follows from the standard parabolic estimate. □

Theorem 5.14 (Regular fibration)**.** *Let $(M_j,g_j)\xrightarrow{GH}(M_\infty,g_\infty)$ be a converging sequence that satisfies $|\nabla^k\operatorname{Rm}_{g_j}|\leq C_k$ and M_∞ is smooth. Let $\mathcal{Q}$ be a connected compact domain in M_∞ and let $\partial\mathcal{Q}$ be smooth. Then we can find $j_0=j_0(\mathcal{Q})>0$ and a sequence $\tau_j\to 0$, such that for all $j\geq j_0$, there exist a compact connected domain $\mathcal{Q}_j\subset X_j^4$ with smooth boundary, together with a smooth fiber bundle map $F_j:\mathcal{Q}_j\to\mathcal{Q}$ such that the following properties hold.*

1. *$F_j:\mathcal{Q}_j\to\mathcal{Q}$ is a τ_j-Gromov-Hausdorff approximation;*
2. *For any $k\in\mathbb{Z}_+$, there exists $C_k>0$ such that for all $j\geq j_0$, we have*
$$|\nabla^k F_j|\leq C_k;\tag{5.19}$$
3. *There exists a uniform constant $C_0>0$ such that for all $q\in\mathcal{Q}$ and $j\geq j_0$, we have*
$$|\operatorname{II}_{F_j^{-1}(q)}|\leq C_0,$$
where $\operatorname{II}_{F_j^{-1}(q)}$ denotes the second fundamental form of the fiber $F_j^{-1}(q)$ at $q\in\mathcal{Q}$.
4. *F_j is an almost Riemannian submersion, in the sense that for any vector v orthogonal to the fiber of F_j, we have*
$$(1-\tau_j)|v|_{g_j}\leq|dF_j(v)|_{g_\infty}\leq(1+\tau_j)|v|_{g_j};\tag{5.20}$$

(5) *There are flat connections with parallel torsion on* $F_j^{-1}(q)$*, which depend smoothly on* $q \in \mathcal{Q}$*, such that each fiber of* F_j *affine diffeomorphic to an infranilmanifold* $\Gamma \setminus N$*, where* N *is a simply-connected nilpotent Lie group, and* Γ *is a cocompact subgroup of* $N_L \rtimes \mathrm{Aut}(N)$ *with* $N_L \simeq N$ *acting on* N *by left translation. Moreover, the structure group of the fibration is reduced to* $((\mathfrak{Z}(N) \cap \Gamma) \setminus \mathfrak{Z}(N)) \rtimes \mathrm{Aut}(\Gamma) \subset \mathit{Aff}(\Gamma \setminus N)$*;*
(6) $\Lambda \equiv \Gamma \cap N_L$ *is normal in* Γ *with* $\#(\Lambda \setminus \Gamma) \leq w_0$ *for some constant* w_0 *independent of* i*.*

Proof of Theorem 5.14. We adopt the notation in the setting of Theorem 5.14. By a simple rescaling, we can assume $\mathrm{injrad}_{g_\infty}(q) \geq 10$ for any $q \in \mathcal{Q}$. To begin with, we fixe $\epsilon > 0$ sufficiently small, and define the rescaled Riemannian metric $h_\epsilon \equiv \epsilon^{-1} \cdot g_\infty$ on $\mathcal{R}$. Throughout the proof we will denote by $\tau(\epsilon)$ a general function of ϵ satisfying $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$. Now for any $q \in \mathcal{Q}$, there is a harmonic coordinate system

$$\varpi_q \equiv (w_1, \ldots, w_d) : B_5^{h_\epsilon}(q) \to B_5(0^d),$$

such that

(i) $\Delta_{h_\epsilon} w_\alpha = 0$ for any $1 \leq \alpha \leq d$,
(ii) $|h_{\epsilon,\alpha\beta} - \delta_{\alpha\beta}|_{C^0(B_4(q))} + |\partial_{w_\gamma} h_{\epsilon,\alpha\beta}|_{C^0(B_4(q))} \leq \tau(\epsilon)$,

where $h_{\epsilon,\alpha\beta} \equiv h_\epsilon(\nabla_{h_\epsilon} w_\alpha, \nabla_{h_\epsilon} w_\beta)$. In particular, we have $d_{GH}(B_4^{h_\epsilon}(q), B_4(0^d)) < \tau(\epsilon)$, where $0^d \in \mathbb{R}^d$.

In the following, we will also work with the rescaled metrics $h_i \equiv \epsilon^{-1} \cdot g_i$ on X_i^4. Unless otherwise specified, the metric balls below will be measured in terms of h_i and h_ϵ, respectively.

We will prove the theorem in three steps. In the first step, using the harmonic splitting map, we will construct local fiber bundle maps over every ball in $\mathcal{Q}$ which looks like a ball in $\mathbb{R}^d$. The second step is to glue the local fiber bundle maps by the well-behaved partition of unity. In the last step, we will show the desired estimates and identify the topology of the collapsing fibers.

Step 1 (construction of local fiber bundles).

Let $\{\underline{q}_\ell\}_{\ell=1}^N$ be a subset of $\mathcal{Q}$ such that $\mathcal{Q} \subset \bigcup_{\ell=1}^N B_1(q_\ell) \subset \mathcal{R}$, and for all $1 \leq \ell, \ell' \leq N$ with $\ell \neq \ell'$, we have $d_{h_\epsilon}(\underline{q}_\ell, \underline{q}_{\ell'}) > \frac{1}{2}$. For every $1 \leq \ell \leq N$, let $q_{i,\ell} \in X_i^4$ such that $(B_4(q_{i,\ell}), h_i) \xrightarrow{GH} (B_4(\underline{q}_\ell), h_\epsilon)$. Then for any sufficiently large i, we have $d_{GH}(B_4(q_{i,\ell}), B_4(0^d)) < 2\tau(\epsilon)$. Then there exists a harmonic map

$$\Phi_{i,\ell}^* = (u_{i,\ell}^{(1)}, \ldots, u_{i,\ell}^{(k)}) : B_3(q_{i,\ell}) \to \mathbb{R}^d$$

which satisfies the following integral estimates

$$\sum_{\alpha,\beta=1}^{d} \fint_{B_3(q_{i,\ell})} |h_i(\nabla_{h_i} u_{i,\ell}^{(\alpha)}, \nabla_{h_i} u_{i,\ell}^{(\beta)}) - \delta_{\alpha\beta}| + \sum_{\alpha=1}^{d} \fint_{B_3(q_{i,\ell})} |\nabla^2 u_{i,\ell}^{(\alpha)}|_{h_i}^2 \leq \tau(\epsilon).$$

Since $(B_4(q_{i,\ell}), h_i)$ is collapsing with uniformly bounded geometry, the above integral estimate can be strengthened to the following pointwise estimate on $B_2(q_{i,\ell})$:

$$\sum_{\alpha,\beta=1}^{k} |g_i(\nabla u_{i,\ell}^{(\alpha)}, \nabla u_{i,\ell}^{(\beta)}) - \delta_{\alpha\beta}| + \sum_{\alpha=1}^{k} |\nabla^2 u_{i,\ell}^{(\alpha)}|^2 \leq \tau(\epsilon).$$

This implies that, for every $1 \leq \ell \leq N$, the composition

$$\Phi_{i,\ell} \equiv (\varpi_\ell)^{-1} \circ \Phi_{i,\ell}^* : B_2(q_{i,\ell}) \to B_2(\underline{q}_\ell)$$

is a fiber bundle map, where the diffeomorphism $\varpi_\ell : B_2(\underline{q}_\ell) \to B_2(0^d)$ is given by the harmonic coordinate system at $\underline{q}_\ell$. Moreover, $\Phi_{i,l}$ is a $\tau(\epsilon)$-Gromov-Hausdorff approximation for all i large.

Step 2 (gluing local bundle maps).

Let us take domains with smooth boundary $\mathcal{Q}_i \subset \bigcup_{\ell=1}^{N} B_2(q_{i,\ell})$ such that $(\mathcal{Q}_i, h_i) \xrightarrow{GH} (\mathcal{Q}, h_\epsilon)$. We will glue the above local harmonic maps to obtain a fiber bundle map $F_i : \mathcal{Q}_i \to \mathcal{Q}$.

For every $1 \leq \ell \leq N$, let ψ_ℓ be the good cut-off function in Lemma 5.13 such that

$$\psi_\ell(y) = \begin{cases} 1, & y \in B_1(q_{i,\ell}), \\ 0, & y \in X_i^4 \setminus B_2(q_{i,\ell}), \end{cases}$$

and for all $m \in \mathbb{Z}_+$, $|\nabla^m \psi_\ell| \leq C_m$ holds everywhere on M_i^4. Then we take the partition of unity subordinate to the cover $\{B_2(q_{i,\ell})\}_{\ell=1}^N$ of $\mathcal{Q}_i$ given by

$$\phi_\ell \equiv \psi_\ell (\sum_{\ell=1}^{N} \psi_\ell)^{-1}.$$

It follows from volume comparison that the multiplicity in the above cover is bounded by some absolute constant $Q_0 > 0$. We denote $\mathcal{B}_i \equiv \bigcup_{\ell=1}^{N} B_2(q_{i,\ell})$ and $\mathcal{B}_\infty \equiv \bigcup_{\ell=1}^{N} B_2(\underline{q}_\ell)$. For any $1 \leq \ell \leq N$, we define

$$\mathfrak{d}_\epsilon(\underline{x}, \underline{y}) \equiv \sum_{\alpha=1}^{d} |w_\alpha(\underline{x}) - w_\alpha(\underline{y})|^2,$$

which is determined by the harmonic coordinate system on $B_2(q_\ell)$. It follows from the estimates on the harmonic coordinates that $|\mathfrak{d}_\epsilon - d_{h_\epsilon}^2| \leq \tau(\epsilon)$ holds on $B_2(\underline{q}_\ell)$. Then let us define the energy function $\mathfrak{E} : \mathcal{B}_i \times \mathcal{B}_\infty \to (0, \infty)$ by

$$\mathfrak{E}(q_i, q_\infty) \equiv \frac{1}{2}\sum_{\ell=1}^{N} \phi_\ell(q_i) \cdot \mathfrak{d}_\epsilon\Big(\Phi_{i,\ell}(q_i), q_\infty\Big).$$

By convexity, for any $q_i \in \mathcal{B}_i$, the function $\mathfrak{E}(q_i, \cdot) : \mathcal{B}_\infty \to [0, \infty)$ has a unique minimum $\mathfrak{z}(q_i)$. It is straightforward to verify that for any $q_i \in B_2(q_{i,\ell})$,

$$d_{h_\epsilon}\Big(\mathfrak{z}(q_i), \Phi_{i,\ell}(q_i)\Big) < \tau(\epsilon),$$

and

$$\Big|h_i(\nabla_{h_i} F_i^{(\alpha)}, \nabla_{h_i} F_i^{(\beta)}) - h_i(\nabla_{h_i} u_{i,\ell}^{(\alpha)}, \nabla_{h_i} u_{i,\ell}^{(\beta)})\Big| \leq \tau(\epsilon).$$

Then we define the map

$$F_i : \mathcal{B}_i \to \mathcal{B}_\infty, \quad q_i \mapsto \mathfrak{z}(q_i), \quad \forall \ q_i \in \mathcal{B}_i.$$

Combining the above estimates on the harmonic splitting maps, harmonic coordinates, as well as the good cut-off functions, we conclude that F_i is a fiber bundle map. Moreover, $F_i : \mathcal{B}_i \to \mathcal{B}_\infty$ is a $\tau(\epsilon)$-Gromov-Hausdorff approximation. Therefore, the proof of item (1) is complete by taking $\epsilon \to 0$ and $i \to \infty$.

Step 3 (proof of the higher order regularity estimates).

In this step, we will rescale everything back to the original metrics g_i and g_∞, respectively. Notice that the uniform estimates for the higher derivatives of the good cut-off functions (constructed in Lemma 5.13) hold in our case, and the higher order estimates for the splitting maps $\Phi_{i,\ell}$ and the harmonic coordinates on $\mathcal{Q}$ hold as well. Then we obtain the pointwise estimate on the second fundamental form in item (2) and the higher order estimate $\nabla^k F_i$ in item (3). We skip the details.

We will prove item (4) by contradiction. Assume that there exist a sequence $\eta_i \to 0$, a constant $\tau_0 > 0$, and a sequence of bundle maps $F_i : \mathcal{Q}_i \to \mathcal{Q}$ which are η_i-Gromov-Hausdorff approximations such that for all sufficiently large i,

$$\Big|\frac{|dF_i(v)|_{g_\infty}}{|v|_{g_i}} - 1\Big| \geq \tau_0 \tag{5.21}$$

holds for a sequence of vectors $v_i \in T_{x_i}\mathcal{Q}_i$ orthogonal to the fiber of F_i. We assume $|v|_{g_i} = 1$. We take the universal cover of $B_{r_0}(x_i)$ for some sufficiently small constant $r_0 > 0$, which gives the following equivariant C^k-convergence for any $k \in \mathbb{Z}_+$:

$$\begin{tikzcd}
(\widetilde{B_{r_0}(x_i)}, \tilde{g}_i, \Gamma_i, \tilde{x}_i) \arrow[r, "C^k"] \arrow[d, "\pi_i"'] & (\widetilde{B}_\infty, \tilde{g}_\infty, \Gamma_\infty, \tilde{x}_\infty) \arrow[d, "\pi_\infty"] \\
(B_{r_0}(x_i), g_i) \arrow[r, "GH"] & (B_{r_0}(x_\infty), g_\infty),
\end{tikzcd}$$

where $\pi_i : (\widetilde{B_{r_0}(x_i)}, \tilde{x}_i) \to (B_{r_0}(x_i), x_i)$ is the Riemannian universal cover with $\pi_i(\tilde{x}_i) = x_i$, $\Gamma_i \equiv \pi_1(B_{r_0}(x_i))$, and Γ_∞ is a closed subgroup in $\mathrm{Isom}_{\tilde{g}_\infty}(\widetilde{B}_\infty)$. The above diagram of equivariant convergence implies that $\pi_\infty : \widetilde{B}_\infty \longrightarrow B_{r_0}(x_\infty) = \widetilde{B}_\infty/\Gamma_\infty$ is a Riemannian submersion. Let $\widetilde{F}_i \equiv F_i \circ \pi_i$ and $\tilde{v}_i$ be the lift of v_i to $\tilde{x}_i$. Then the C^k convergence implies that $\widetilde{F}_i$ converges to π_∞, and $\tilde{v}_i$ converges to a limiting vector $\tilde{v}_\infty$ with $|d\pi_\infty(v_\infty)|_{g_\infty} = 1$. This contradicts (5.21), which completes the proof of item (4).

Based on the Gromov-Hausdorff estimate in item (1) and the second fundamental form estimate in item (3), we can conclude that all the fibers of F_i are almost flat manifolds in the sense that

$$\mathrm{diam}(F_i^{-1}(q))^2 \cdot |\sec_{F_i^{-1}(q)}| < \tau(\epsilon), \quad \forall q \in \mathcal{Q},$$

for any sufficiently large i. If ϵ is chosen sufficiently small, then item (5) and (6) follows from Gromov and Ruh's theorems on the almost flat manifolds and Fukaya's fibration theorem; see [Gro81, Ruh82, Fuk89, CFG92]. $\square$

6. Selected Results in Collapsing Einstein Manifolds

Some selected topics:

(1) ϵ-regularity in 4D; Margulis lemma
(2) Quantitative Margulis Lemma; ϵ-regularity in higher dimensions
(3) First Betti number and almost nonnegative Ricci curvature; parametrized version
(4) Collapsing Einstein manifolds with special holonomy; examples

References

[CFG92] Jeff Cheeger, Kenji Fukaya, and Mikhael Gromov, *Nilpotent structures and invariant metrics on collapsed manifolds*, J. Amer. Math. Soc. **5** (1992), no. 2, 327–372.

[Fuk89] K. Fukaya, *Collapsing Riemannian manifolds to ones with lower dimension. II*, J. Math. Soc. Japan **41** (1989), no. 2, 333–356.

[Gro81] Mikhael Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques [Mathematical Texts], vol. 1, CEDIC, Paris, 1981, Edited by J. Lafontaine and P. Pansu.

[Ruh82] E. Ruh, *Almost flat manifolds*, J. Differential Geometry **17** (1982), no. 1, 1–14.

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