Putnam Club 2018
Sequences

Determining closed form expressions for sequences

Useful concepts:

- Pattern recognition and induction.
- Greatest integer function:
$$\lfloor x \rfloor := \max\{z \in \mathbb{Z} : z \leq x\}$$

1. Consider the sequence (a_i) given by
$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right)$$
where $m \geq n \geq 0$. Find a formula for a_n if $a_1 = 1$.

2. Find a formula for the general term of the sequence
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots.$$

Recursive Sequences

Useful Concept: Characteristic Equation. Let $x_n = \sum_{i=1}^{k} a_i x_{n-i}$ for some $k \leq n$. Interpret the equation as the component description of a matrix vector product to find a matrix A so that $v_n = A^n v_0$ for some vector v_n which describes the sequence x_n. The characteristic polynomial of A is:
$$P(\lambda) = \sum_{i=0}^{k} -a_i \lambda^{k-i}$$
where $a_0 = -1$.

Let $\{\lambda_i\}_{i=1}^{t}$ be the roots of P (the eigenvalues of A) with multiplicity m_i. Then
$$x_n = \sum_{i=1}^{t} \sum_{j=0}^{m_i - 1} c_{ij} \binom{n}{j} \lambda_i^{n-j}$$
for some constants c_{ij}. In the case that $m_i = 1$ this becomes:
$$x_n = \sum_{i=1}^{k} c_i \lambda_i^n.$$

1. Find the general term of the sequence given by $x_0 = 3, x_1 = 4$, and
$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-2}$$
for $n \geq 2$.

2. Consider the sequences
$$a_0 = 1, \quad a_{n+1} = \frac{3a_n + \sqrt{5a_n^2 - 4}}{2}$$
$$b_0 = 0, \quad b_{n+1} = a_n - b_n.$$
Prove that $(a_n)^2 = b_{2n+1}$ for all n.

Limits of sequences

Useful concepts:

- Classic analytic definition of limit: For all $\varepsilon > 0$ there is an N so that $n \geq N \Rightarrow |x_n - L| < \varepsilon$. Consider how this changes for a limit equal to infinity.
- Squeeze theorem: $a_n \leq b_n \leq c_n$, $a_n \to L$, $c_n \to L$ then $b_n \to L$. Consider also a version which shows that a limit is infinite.
- Bounded and monotone means convergent.
- Cauchy criterion: Let x_n be a sequence in a complete metric space (e.g., $\mathbb{R}^n$) then x_n is convergent if and only if for all $\varepsilon > 0$ there is N so that $n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon$.
- Cesàro-Stolz theorem (discrete analog to L'Hôpital): Let x_n and y_n be two real sequences with y_n positive, increasing, and unbounded. Then
$$\frac{x_{n+1} - x_n}{y_{n+1} - y_n} \to L \Rightarrow \frac{x_n}{y_n} \to L.$$
- Nested intervals: Let I_k be a sequence of closed intervals with $I_k \supset I_{k+1}$ and diameter of I_k converging to zero. Then $\bigcap I_k$ is one point.

1. Let x_n be a sequence with the property that $x_{x_n} = n^4$ for all $n \geq 1$. Prove that $x_n \to \infty$.
2. Prove that
$$n^2 \int_0^{\frac{1}{n}} x^{x+1} \, \mathrm{d}x \to \frac{1}{2}.$$
3. Let a be a positive real number and x_n a sequence with $x_1 = a$ and
$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} k x_k.$$
Find the limit of x_n.
4. Show that
$$a_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1)$$
is convergent.
5. Let
$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{b_n + c_n}{2}, \quad c_{n+1} = \frac{c_n + a_n}{2}$$
Assuming given values for a_0, b_0, c_0 show that all three sequences converge and find their limits.
6. Let t and ε be real numbers with $|\varepsilon| < 1$. Then $x - \varepsilon \sin x = t$ has a unique real solution.
7. Let c, x_0 be fixed positive real numbers. Then
$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{c}{x_{n-1}}\right) \to \sqrt{c}.$$
8. Let k be an ineger larger than one. Suppose $a_0 > 0$ and define:
$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}.$$
Evaluate
$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$
9. Let $f : [a, b] \to [a, b]$ be an increasing function. Show that there is $c \in [a, b]$ so that $f(c) = c$.