Let A, B be 2×2 matrices with integer entries, such that $AB = BA$ and $\det B = 1$. Prove that if $\det(A^3 + B^3) = 1$, then $A^2 = \mathcal{O}_2$.

Consider the $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = 1$ if $j - i \equiv 1 \pmod{n}$ and $a_{ij} = 0$ otherwise. For real numbers a and b find the eigenvalues of $aA + bA^t$.

Let A be an $n \times n$ matrix. Prove that there exists an $n \times n$ matrix B such that $ABA = A$.

Consider the angle formed by two half-lines in three-dimensional space. Prove that the average of the measure of the projection of the angle onto all possible planes in the space is equal to the angle.

A linear map A on the n-dimensional vector space V is called an involution if $A^2 = \mathcal{I}$.

(a) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

(b) Find the maximal number of distinct pairwise commuting involutions.

Let A be a 3×3 real matrix such that the vectors Au and u are orthogonal for each column vector $u \in \mathbb{R}^3$. Prove that

(a) $A^t = -A$, where A^t denotes the transpose of the matrix A;

(b) there exists a vector $v \in \mathbb{R}^3$ such that $Au = v \times u$ for every $u \in \mathbb{R}^3$.

Denote by $M_n(\mathbb{R})$ the set of $n \times n$ matrices with real entries and let $f : M_n(\mathbb{R}) \to \mathbb{R}$ be a linear function. Prove that there exists a unique matrix $C \in M_n(\mathbb{R})$ such that $f(A) = \operatorname{tr}(AC)$ for all $A \in M_n(\mathbb{R})$. In addition, if $f(AB) = f(BA)$ for all matrices A and B, prove that there exists $\lambda \in \mathbb{R}$ such that $f(A) = \lambda \operatorname{tr} A$ for any matrix A.

Let U and V be isometric linear transformations of $\mathbb{R}^n$, $n \geq 1$, with the property that $\|Ux - x\| \leq \frac{1}{2}$ and $\|Vx - x\| \leq \frac{1}{2}$ for all $x \in \mathbb{R}^n$ with $\|x\| = 1$. Prove that

$$\|UVU^{-1}V^{-1}x - x\| \leq \frac{1}{2},$$

for all $x \in \mathbb{R}^n$ with $\|x\| = 1$.

For an $n \times n$ matrix A denote by $\phi_k(A)$ the symmetric polynomial in the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A,

$$\phi_k(A) = \sum_{i_1 i_2 \cdots i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \quad k = 1, 2, \ldots, n.$$