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Convergence and Continuity

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1 Functions: a reminder of classical results and problems

Continuity (definition). For x_0 an accumulation point of the domain of the definitipn of a function f, we say that $\lim_{x \to x_0} f(x) = L$ if for every neighborhood V of L, there is a neighborhood U of x_0 such that $f(U) \subset V$. A function that is continuous at every point of its domain is simply called continuous.

Peanos theorem. There exists a continuous surjection $\phi : [0,1] \to [0,1] \times [0,1]$

Darboux property. A real-valued function f defined on an interval is said to have the intermediate value property (or the Darboux property) if for every $a < b$ in the interval and for every λ between $f(a)$ and $f(b)$, there exists c between a and b such that $f(c) = \lambda$. Equivalently, a real-valued function has the intermediate property if it maps intervals to intervals. The higher-dimensional analogue requires the function to map connected sets to connected sets. Continuous functions and derivatives of functions are known to have this property, although the class of functions with the intermediate value property is considerably larger.

Important property of continuous functions. A general property that the image through a continuous map of a connected set is connected.

Derivatives (definition). A function f defined in an open interval containing the point x_0 is called differentiable at x_0 if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. In this case, the limit is called the derivative of f and is denoted by $f'(x_0)$. If the derivative is defined at every point of the domain of f, then f is simply called differentiable.

The mean value theorem. If $f : [a,b] \to \mathbb{R}$ is a function that is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's theorem. If $f : [a, b] \to \mathbb{R}$ is a function that is continuous on $[a, b]$ and differentiable on (a, b), and satisfies $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Example. Prove that the Legendre polynomial

$$P_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n$$

has n distinct zeros in the interval $(-1, 1)$.

Cauchy's theorem. If $f, g : [a, b] \to \mathbb{R}$ are two functions, continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c).$$

Hölder's inequality. If $x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_n, p$ and q are positive numbers with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^p\right)^{1/q}$$

with equality if and only if the two vectors $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are parallel.

Jensen's inequality. For a convex function f let $x_1 \ldots$ le points in its domain and let λ_i, $i \geq 1$ be positive numbers such that $\sum_i \lambda_i = 1$. Then

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots \lambda_n f(x_n).$$

If is f nowhere linear and the x's are not all equal, then the inequality is strict. The inequality is reversed for a concave function.

1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that

   $$\int_0^1 f(x)\, dx = \frac{\pi}{4}.$$

   Prove that there exists $x_0 \in (0, 1)$ such that

   $$\frac{1}{1 + x_0} < f(x_0) < \frac{1}{2x_0}.$$

2. Prove that any convex polygonal surface can be divided by two perpendicular lines into four regions of equal area.

3. Prove that for any natural number $n \geq 2$ and any $|x| \leq 1$,

$$(1+x)^n + (1-x)^n \leq 2^n.$$

4. Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.) Find

$$\int_0^1 f(x)\, dx.$$

5. Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\, dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\, dx$ be?

6. Let $f : [0,1] \to \mathbb{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [0,1]$. Suppose also that for each rational number $r \in [0,1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \ldots, I_n$ such that f is a linear function on each I_i and $[0,1] = \bigcup_{i=1}^n I_i$.

7. Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx$ diverges.

2 Sequences: a reminder of classical results and more problems

Monitonicity. Every bounded monotone real sequence $a_1, a_2, \ldots$ converges to a limit.

Cauchy sequence (definition). A sequence $a_1, a_2, \ldots$ is called *Cauchy sequence* if for every $\epsilon > 0$, there exists a positive integer N such that for all $i, j > N$ we have

$$|a_i - a_j| < \epsilon.$$

The real and complex number systems have the property that every Cauchy sequence converges to a limit, which is a number in the system.

Absolute convergence. Let $z_1, z_2, \ldots$ be a sequene of complex numbers, for which $\sum_i |z_i|$ converges. Then $\sum_i z_i$ converges as well.

Abel summation. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences, and let $B_k := \sum_{i=1}^k b_i$ for every k. Then

$$\sum_{i=1}^n a_i b_i = a_n B_n - \sum_{i=1}^{n-1} B_i (a_{i+1} - a_i).$$

Classical problem. Prove that the sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots$ converges, and determine its limit. This sequence is often denoted as

$$\sqrt{7+\sqrt{7+\sqrt{7+\cdots}}}.$$

1 Show that if $\sum_{i=1}^{\infty} a_i^2$ and $\sum_{i=1}^{\infty} b_i^2$ both converge, then so does $\sum_{i=1}^{\infty}(a_i - b_i)^p$, for every $p \geq 2$.

2 Let $a_1, a_2, \ldots$ be positive integers such that $\sum(1/a_i)$ converges. For each n, let b_n denotes the number of positive integers i for which $a_i \leq n$. Prove that $\lim_{n\to\infty}(b_n/n) = 0$.

3 Let $a_1, a_2, \ldots$ be a sequence of real numbers for which $\sum_{i=1}^{\infty} a_i$ converges. Show that the sum $\sum_{i=1}^{\infty}(a_i/i)$ also converges.

4 Let $\{a_i\}$ be a monotonically decreasing sequence of positive real numbers, for which $\sum_{i=1}^{\infty} a_i$ converges. Show that

$$\sum_{i=1}^{\infty} i(a_i - a_{i+1})$$

also converges.

5 Let α be an arbitrary real number. Define $a_1 := \alpha$, and for all $n \geq 1$, let

$$a_{n+1} = \cos(a_n).$$

Prove that a_n converges to a limit, and that this limit dies not depend on α.

6 Prove that the sequence $\sqrt{7}, \sqrt{7-\sqrt{7}}, \sqrt{7-\sqrt{7+\sqrt{7}}}, \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}, \ldots$ converges, and determine its limit.

7 Let $z_1, z_2, \ldots$ be nonzero complex numbers with the property that $|z_i - z_j| > 1$ for all pairs (i, j). Prove that $\sum_i (1/z_i^3)$ converges.

8 Let $\{a_i\}$ be a sequence of positive real numbers. Show that $\limsup \left(\frac{a_1+a_{n+1}}{a_n}\right)^n \geq e$.

9 Let $a_1, a_2, \ldots$ be a sequence of positive real numbers for which $\sum_{i=1}^{\infty}(1/a_i)$ converges. For every n, let $b_n = \frac{a_1+\cdots+a_n}{n}$. Show that $\sum_{i=1}^{\infty}(1/b_n$ also converges.

10. Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$