Putnam Club. Fall 2020

Problem session for October 28. Polynomials. Solutions.

1. a, b, and c are the three roots of the polynomial $x^3 - 3x^2 + 1$. Find $a^3 + b^3 + c^3$.

   Solution. Using $a^3 = 3a^2 - 1$ and similar relations for b, c, we get

   $$a^3 + b^3 + c^3 = 3(a^2 + b^2 + c^2) - 3 = 3\big((a + b + c)^2 - 2(ab + ac + bc)\big) - 3.$$

   Viete relations gives $a + b + c = 3$, and $ab + ac + bc = 0$, thus

   $$a^3 + b^3 + c^3 = 3 \cdot 3^2 - 3 = 24.$$

2. Find all polynomials satisfying the functional equation

   $$xP(x - 1) = (x - 20)P(x).$$

   Solution. Substituting $x = 0$, we get that 0 is a root of $P(x)$, so $P(x) = xP_1(x)$. Therefore

   $$x(x - 1)P_1(x - 1) = (x - 20)xP_1(x) \Rightarrow (x - 1)P_1(x - 1) = (x - 20)P_1(x).$$

   Substituting $x = 1$, we get that 1 is a root of $P_1(x)$, so $P_1(x) = (x - 1)P_2(x)$. Therefore

   $$(x - 1)(x - 2)P_2(x - 1) = (x - 20)(x - 1)P_2(x) \Rightarrow (x - 2)P_2(x - 1) = (x - 20)P_2(x).$$

   Repeating similar arguments we can get

   $$\begin{aligned} &(x - 3)P_3(x - 1) = (x - 20)P_3(x), \\ &\dots\dots\dots \\ &(x - 20)P_{20}(x - 1) = (x - 20)P_{20}(x), \end{aligned}$$

   and $P(x) = x(x - 1)\dots(x - 19)P_{20}(x)$. Thus, since equation

   $$P_{20}(x - 1) = P_{20}(x)$$

   can be valid for any x only for $P_{20}(x) = const$, we get finally

   $$P(x) = Cx(x - 1)\dots(x - 20).$$

3. Let $P(x)$ and $Q(x)$ be monic polynomials of degree 10. It is known, that the equation $P(x) = Q(x)$ has no real roots. Prove that the equation $P(x + 1) = Q(x - 1)$ has at least one real root.

   Solution. Recall that any polynomial of odd degree has a real root. Since $P(x) - Q(x)$ is a polynomial of the degree at most 9, and it does not have any real roots, it must be of the degree at most 8, i.e. coefficients at x^9 in $P(x)$ and $Q(x)$ are equal. Hence the coefficient at x^9 in $P(x + 1) - Q(x - 1)$ is the same that the coefficient at x^9 in $(x + 1)^{10} - (x - 1)^{10}$, i.e. 20. Therefore $P(x + 1) - Q(x - 1)$ is of the degree 9, and so it has at least one real root.

4. Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$, counted without multiplicities.

   Solution. Let $P(x) = (x-x_1)(x-x_2)\dots(x-x_k)Q(x)$, where $x_1,\dots, x_k$ are all different roots of $P(x)$, and $Q(x)$ is everything left in factorization of P. Then

   $$P(P(x)) = (P(x) - x_1)\dots(P(x) - x_k)Q(P(x)).$$

   Notice that $P(x) - x_i$ has an odd degree so it must have solution for all i. In additions, this solutions should be different since all x_i are different. That finishes the proof.

5. Let $P(x)$ be a polynomial and a_1, a_2, b_1, b_2 be some real numbers. Assume that for any real x we have $P(a_1x + b_1) + P(a_2x + b_2) = P(x)$. Prove that $P(x)$ has at least one real root.

   Solution. Notice that if $a_1 \neq 1$, then there exists x_1 such that $a_1x_1 + b_1 = x_1$, and thus substitution $x = x_1$ gives $P(a_2x_1 + b_2) = 0$. Therefore $a_1 = 1$. Similarly $a_2 = 1$. Now look at the top coefficient: if $P(x) = a_nx^n + \ldots + a_0$, then

   $$P(x + b_1) + P(x + b_2) = 2a_nx^n + \ldots,$$

   so it cannot be equal to $P(x)$.

6. Let $P(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$, where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then r_1r_2 is a rational number.

   Solution. The first Viete relation gives

   $$r_1 + r_2 + r_3 + r_4 = -b/a,$$

   so $r_3 + r_4$ is rational. Also,

   $$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = c/a.$$

   Therefore,

   $$r_1r_2 + r_3r_4 = c/a - (r_1 + r_2)(r_3 + r_4).$$

   Finally,

   $$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = -d/a,$$

   which is equivalent to

   $$(r_1 + r_2)r_3r_4 + (r_3 + r_4)r_1r_2 = -d/a.$$

   We observe that the products r_1r_2 and r_3r_4 satisfy the linear system of equations

   $$\begin{aligned} \alpha x + \beta y &= u, \\ \gamma x + \delta y &= v, \end{aligned}$$

   where $\alpha = 1$, $\beta = 1$, $\gamma = r_3 + r_4$, $\delta = r_1 + r_2$, $u = c/a?(r_1 + r_2)(r_3 + r_4)$, $v = -d/a$. Because $r_1 + r_2 = r_3 + r_4$, this system has a unique solution; this solution is rational. Hence both r_1r_2 and r_3r_4 are rational, and the problem is solved.

7. Let $P(x)$ be a polynomial with integer coefficients such that $|P(3)| = |P(7)| = 1$. Prove that $P(x)$ does not have any integer roots.

   Solution. We are going to use $x - y | P(x) - P(y)$. Let $P(a) = 0$. Then

   $$a - 3 | P(a) - P(3) = \pm 1, \quad a - 7 | P(a) - P(7) = \pm 1,$$

   hence

   $$a - 3 = \pm 1, \quad a - 7 = \pm 1,$$

   which is impossible.

8. Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros $x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n$ are real. Prove that

   $$P'\Big(\frac{x_1 + x_2}{2}\Big) \cdot P'\Big(\frac{x_{n-1} + x_n}{2}\Big) \neq 0.$$

   Solution. We are going to use

   $$\frac{P'(x)}{P(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \ldots + \frac{1}{x - x_n}.$$

   If we substitute $x_0 = (x_1 + x_2)/2$, we obtain

   $$\frac{P'(x_0)}{P(x_0)} = \frac{1}{x_0 - x_1} + \frac{1}{x_0 - x_2} + \ldots + \frac{1}{x_0 - x_n}.$$

   Since

   $$\frac{1}{x_0 - x_1} + \frac{1}{x_0 - x_2} = \frac{2}{x_1 - x_2} + \frac{2}{x_2 - x_1} = 0,$$

   and $x_0 - x_i < 0$ for all $i = 3, \ldots, n$, we get

   $$\frac{P'(x_0)}{P(x_0)} < 0,$$

   thus $P'\Big(\frac{x_1 + x_2}{2}\Big) \neq 0$. Similarly we can prove $P'\Big(\frac{x_{n-1} + x_n}{2}\Big) \neq 0$.

9. Let $p(x)$ be a real polynomial that is non-negative for all real x. Prove that for some k, there are real polynomials $f_1(x), \ldots, f_k(x)$ such that $p(x) = f_1(x)^2 + f_2(x)^2 + \ldots + f_k(x)^2$.

   Solution. Since $p(x) \geq 0$, any real root c_i of $p(x)$ must have even multiplicity. All complex roots of $p(x)$ must be in conjugate pairs, and

   $$(x - (a + ib))(x - (a - ib)) = (x - a)^2 + b^2,$$

   so we can factorize $p(x)$ as

   $$p(x) = (x - c_1)^{2k_1} \cdot \ldots \cdot (x - c_p)^{2k_p} \cdot \big((x - a_1)^2 + b_1^2\big) \cdot \ldots \cdot \big((x - a_q)^2 + b_q^2\big).$$

   Expanding parentheses in this expression we get $p(x) = f_1(x)^2 + f_2(x)^2 + \ldots + f_k(x)^2$.

10. Suppose $P(x)$ is a polynomial of degree 3. A valid operation consists of replacing the current polynomial $Q(x)$ by either $Q(x)+Q'(x)$ or $Q(x)-Q'(x)$. Prove that a sequence of valid operations starting with $P(x)$ will never result in getting the same polynomial again.

    Solution. Assume the opposite. It is easy to check that the two operations commute. Now let us look on the coefficient at the term with x^2. If the top coefficient of $P(x)$ is a, then it is always a, and thus each operation add to the term with x^2 $\pm 3a$. Therefore the number of operations with plus and with minus must be equal. Hence we can assume that the order of operation is $+,-,+,-,\dots,+,-$. Next, look on the coefficient at x if we perform first the operation with "+", and then with "-":

    $$(Q(x)+Q'(x))-(Q(x)+Q'(x))' = Q(x)-Q''(x) = (ax^3+bx^2+cx+d)-(6ax+2b),$$

    so the coefficient at x is $c-6a$. Since a is a constant, we get the contradiction.

11. Find all polynomials $P(x)$, $Q(x)$ with real coefficients such that

    $$P(x)Q(x+1)-P(x+1)Q(x)=1.$$

    Solution. Suppose p and p satisfy the given equation; note that neither p nor q can be identically zero. By subtracting the equations

    $$p(x)q(x+1)-p(x+1)q(x)=1,$$
    $$p(x-1)q(x)-p(x)q(x-1)=1;$$

    we obtain the equation

    $$p(x)(q(x+1)+q(x-1)) = q(x)(p(x+1)+p(x-1)).$$

    The original equation implies that $p(x)$ and $q(x)$ have no common nonconstant factor, so $p(x)$ divides $p(x+1)+p(x-1)$. Since each of $p(x+1)$ and $p(x-1)$ has the same degree and leading coefficient as $p(x)$, we must have

    $$p(x+1)+p(x-1)=2p(x).$$

    If we define the polynomials $r(x)=p(x+1)-p(x)$, $s(x)=q(x+1)-q(x)$, we have $r(x+1)=r(x)$, and similarly $s(x+1)=s(x)$, so $r(x)$ and $s(x)$ are constants. Consequently, $p(x)=ax+b$, $q(x)=cx+d$. For p and q of this form,

    $$p(x)q(x+1)-p(x+1)q(x)=bc-ad;$$

    so we get a solution if and only if $bc-ad=1$.

12. Find, with proof, for which n there exists a polynomial $P(x)$ of degree n with real coefficients and a polynomial $Q(x)$, such that $P(x^2+x+1)=P(x)Q(x)$.

    Solution. If a is a real root $P(x)$, then $a^2+a+1>a$ is also the real root of P. Continuing this process we can get an infinite increasing sequence of roots of $P(x)$, which is impossible. Hence $P(x)$ cannot have any real roots, thus it must have even degree. It is also easy to check that for any even $n=2k$ the polynomial $(x^2+1)^k$ works.

13. Country Fateland selects its 6-person IMO team in the following way. There are 13 candidates who take team selection test and score $a_1, a_2, \ldots, a_{13}$ points. The scores are assumed to be distinct. Also there is a secret polynomial $P(x)$ with real coefficients that measures "creative potential" of contests. The six students with the highest values of $P(a_i)$ go to the IMO. Unfortunately, Mr. Unfair, the team leader, has pre-determined the team before the test. Find, with proof, the smallest n such that Mr. Unfair can always justify his selection regardless of test results using a polynomial of degree no more than n.

Solution. Because of the interpolation formula, there exists polynomial of degree 12 with any given values in $a_1, \ldots, a_{13}$, so $n = 12$ is enough. Now let us assume that $a_1 < a_2 < \ldots < a_{13}$, and Mr. Unfair wants to select students with results $a_2, a_4, \ldots, a_{12}$. Then if $c = \min\{P(a_2), P(a_4), \ldots, P(a_{12})\} - \varepsilon$, we have $P(a_1) < c$, $P(a_2) > c$, $P(a_3) < c, \ldots,$ $P(a_{12}) > c$, $P(a_{13}) < c$. But this means that polynomial $P(x) - c$ must have at least 12 roots (since on any interval $[a_i, a_{i+1}]$ there is at least one root), thus $P(x)$ has the degree at least 12.