Putnam Club Problem Sheet - November 2

Problem 1. (2019 B3) Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ. (Hint: (1) give a geometric interpretation to P; (2) what is the strongest thing you can say about the eigenvalues of Q?)

Problem 2. (2018 A2) Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

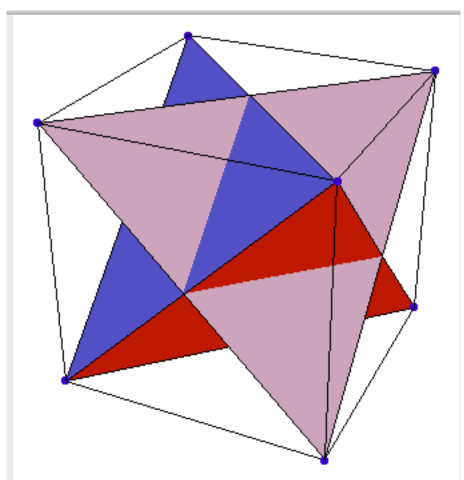
$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M. (Hint: each S_i can be associated to a number between 1 and $2^n - 1$ in binary and then ordered lexicographically.)

Problem 3. (2021 A3) Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates. Hint: consider the following picture.



Problem 4. (2021 B5) Say that an n-by-n matrix $A = (a_{ij})_{1 \le i,j \le n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1, 2, \ldots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \ge 1$. (Hint: write down some 2×2 and 3×3 examples. In all these examples, there is always at least one column vector such that ...?)

Problem 5. (2016 B4)

Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A. (Hint: use the permutation expansion of the determinant).

Problem 6. (2015 A6)

Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries. (Hint: Show that the characteristic polynomials of $A - MX$ and $B - XM$ coincide. Consider starting with the assumption that X is invertible.)

Problem 7. (2015 B3)

Let S be the set of all 2×2 real matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

Problem 8. (2014 A2)

Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

Problem 9. (2012 A5)

Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries of $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \ldots, p^n$ are distinct.