Binomial coefficients and generating functions

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Binomial coefficients

The binomial coefficients $\binom{n}{k}$ counts the number of ways one can choose k objects from given n. They are coefficients in the binomial expansion

$$(x+1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n-1}x + \binom{n}{n}.$$

More explicitly,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Some simple but useful formulas for the binomial coefficients are

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and

$$\binom{n}{k} = \frac{n}{k}\cdot\binom{n-1}{k-1}.$$

Exercise: prove the above two formulas using the binomial expansion.

Generating functions

The terms of a sequence $(a_n)_{n\geq 0}$ can b e combined into a function

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$

called the generating function of the sequence. For example, the finite sequence $\binom{m}{n}$, with m fixed and n varies, gives the function $(x+1)^m$. The generating function for $a_n = \frac{1}{n}$ is $-\ln(1-x)$.

Generating functions provide a method to understand recursive relations of a sequence.

Theorem. *Suppose a_n $(n \geq 0)$ is a sequence satisfying a second-order linear recurrence,*

$$a_n + ua_{n-1} + va_{n-2} = 0.$$

Suppose that the quadratic equation $\lambda^2 + u\lambda + v = 0$ has two distinct roots r_1, r_2. Then

$$a_n = \alpha r_1^n + \beta r_2^n$$

for some real numbers r_1, r_2.

Proof. Let $G(x) = a_0 + a_1x + a_2x^2 + \cdots$ be the generating function of a_n. The recurrence relation of a_n implies that

$$G(x) - a_0 - a_1x + u\big(G(x) - a_0\big) + vx^2G(x) = 0.$$

Solving for $G(x)$, we have

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{1 + ux + vx^2} = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)}.$$

Using partial fractions, we have

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)} = \frac{\alpha}{1 - r_1x} + \frac{\beta}{1 - r_2x} = \sum_{n=1}^{\infty} (\alpha r_1^n + \beta r_2^n)x^n.$$

Therefore, $a_n = \alpha r_1^n + \beta r_2^n$. □

1. Prove the identity

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

2. Give two proofs of the identity

$$\sum_{j=0}^{n} 2^{n-j}\binom{n}{j}\binom{j}{\lfloor j/2 \rfloor} = \binom{2n+1}{n}.$$

More Problems

1. (Putnam and beyond 880) Prove the identity

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

2. (Putnam and beyond 867) Find the general formula for the sequence $(y_n)_{n\geq 0}$ with $y_0 = 1$ and $y_n = ay_{n-1} + b^n$ for $n \geq 1$, where a and b are two fixed distinct real numbers.

3. (Putnam and beyond 872) Prove that the Fibonacci numbers F_n satisfy

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.$$

4. (Putnam and beyond 860) Consider the triangular $n \times n$ matrix

$$A = \begin{Bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{Bmatrix}.$$

Compute the matrix A^k for $k \geq 1$.

5. (Putnam 1992 B-2) Show that the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$ is

$$\sum_{j=0}^{k} \binom{n}{k}\binom{n}{k-2j}.$$

6. Find the generating function of the sequence u_n = number of nonnegative solutions of $2a + 5b = n$.

7. (Putnam 2017 B-3) Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that, if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

8. (Putnam 1962 A-5) Find

$$\binom{n}{1}1^2 + \binom{n}{2}2^2 + \binom{n}{3}3^2 + \cdots + \binom{n}{n}n^2.$$

9. (Putnam 2000 B-5) Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows. The integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there are infinitely many integers N for which

$$S_N = S_0 \cup \{N + a \mid a \in S_0\}.$$

10. (Putnam 2003 A-6) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that

$$s_1, s_2 \in S, s1 \neq s2, \quad \text{and} \quad s1 + s2 = n.$$

Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

11. For positive integer n, denote by $S(n)$ the number of choices of the signs "+" or "−" such that $\pm 1 \pm 2 \pm \cdots \pm n = 0$. Prove that

$$S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos t \cos 2t \cdots \cos nt \, dt.$$