9/28/11 – Counting

1. Determine the number of ordered triples (A_1, A_2, A_3) of sets such that $A_1 \cup A_2 \cup A_3 = \{1, 2, \ldots, 10\}$ and $A_1 \cap A_2 \cap A_3 = \emptyset$. (Putnam 1985)

2. Show that $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

3. **(a)** Show that $\displaystyle\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

(b) Show that $\displaystyle\sum_{k=1}^{n-1}\sum_{j=1}^{n-k} kj\binom{n}{k}\binom{n-k}{j} = n(n-1)3^{n-2}$.

4. How many ways are there to write n as the *ordered* sum of positive integers? (E.g., if $n = 4$, the different sums are 4, 3+1, 1+3, 2+2, 1+1+2, 1+2+1, 2+1+1, and 1+1+1+1.)

5. 14 positive integers are chosen less than 1000. Show that there is some pair of disjoint subsets of these numbers whose sums are equal. (Hint: use counting *and* pigeonhole.)

6. The numbers $1, 2, \ldots, 3n+1$ are written in a random order. What is the probability that the sum of the first k of them (for each $k = 1, 2, \ldots, 3n+1$) is never divisible by 3? (Putnam 2007)