Some Putnam Problems

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1. (2021 A1) A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point (2021,2021)?

2. (2021 A2) For every positive real number x, let

$$g(x) = \lim_{r \to 0} \left((x+1)^{r+1} - x^{r+1} \right)^{\frac{1}{r}}.$$

   Find $\lim_{x \to \infty} \frac{g(x)}{x}$.

3. (2021 B1) Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

4. (2021 A1) How many positive integers N satisfy all of the following three conditions?

   (a) N is divisible by 2020.

   (b) N has at most 2020 decimal digits.

   (c) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

5. (2021 A2) Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}.$$

6. (2019 A1) Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

   where A, B, and C are nonnegative integers.

7. (2018 A1) Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

8. (2016 B1) Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

   (as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

   converges and find its sum.

9. (2021 B5) For $j \in \{1, 2, 3, 4\}$, let z_j be a complex number with $|z_j| = 1$ and $z_j \neq 1$. Prove that

$$3 - z_1 - z_2 - z_3 - z_4 + z_1 z_2 z_3 z_4 \neq 0.$$

10. (2018 A5) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.