10/19/11 – A Grab Bag of Discrete Math

1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S? (Putnam 1992)

2. Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even. (Putnam 2002)

3. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.) (Putnam 2006)

4. Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$. (Putnam 2000)

5. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i? (Putnam 1985)

6. Let

$$
\begin{array}{cccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\
a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n). (Putnam 1985)

7. Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$
c(n, m) = \sum_{j=1}^{m} j^n.
$$

(Putnam 1993)

8. n red and n blue dots, no three of them collinear, lie in the plane. Show that it is possible to draw n line segments, each connecting a red dot and a blue dot, so that no two such segments intersect.

9. n people standing in the plane have a shootout. Simultaneously, all of them draw their guns and shoot dead the person who is standing closest to them. (If there is a tie, they can break it how they choose.) What is the largest number of people who can survive? (You may assume that n is divisible by whatever numbers you like to make things work out nicely.)

10. Find, with proof, all possible integer values of $\dfrac{a^2 + ab + b^2}{ab - 1}$, where a and b are positive integers.