11/2/11 – Calculus (Week 2)

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge? (Putnam 2008)

2. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$? (Putnam 2000)

3. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \le 36(x^2 + y^2).$$

(Putnam 2006)

4. Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\, dx.$$

(Putnam 2005)

5. Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)}.$$

(Putnam 1999)

6. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x. (Putnam 1992)

7. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function, and suppose that there is no $x \in [0, 1]$ such that $f(x) = f'(x) = 0$. Show that f has only finitely many zeros in $[0, 1]$.