Putnam team training 11/11

(1) In a 120×150 rectangle (made out of unit squares joined along their sides), how many unit squares does its diagonal pass through?

(2) Let $\alpha \in \mathbb{R}$. Find all polynomials f of degree at most 3 that commute with $p(x) = x^2 - \alpha$, i.e. such that $f(p(x)) = p(f(x))$ for all x.

(3) Find all integer solutions to $xy + 3x - 5y = -3$.

(4) Find all positive integers x, y such that $4^x + 5 = 9^y$.

(5) Let a and b be positive integers. If $\sqrt{a}$ is irrational, prove that $\sqrt{a} + \sqrt{b}$ is irrational.

(6) On a table there are 100 tokens. Taking turns two players remove 5, 6, 7, 8, 9 or 10 tokens, at their choice. The player that removes the last token wins. Find a winning strategy and determine which player will be the winner.

(7) If $x \neq 0$ prove that

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \cos(x/2^n).$$

(8) Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j \geq 2$. Prove that for any $n \in \mathbb{N}$ all roots of the equation $P_n(x) = x$ are real and distinct.