Putnam Club Week Seven – 30 October 2012

Probability

1. What is the probability $P(n)$ that a random permutation of n objects leaves no object in its original place? What is $\lim_{n\to\infty} P(n)$? (Hint: Use a problem from last week.)

2. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots? (Putnam 2002)

3. You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n. (Putnam 2001)

4. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers. (Putnam 1989)

5. Three real numbers are chosen randomly and uniformly from the interval $[0,1]$. What is the probability that they are the three side lengths of some triangle?

6. There are n people boarding a plane. The first one drops his boarding pass, so he sits in a seat at random. After that, each passenger sits in his or her own seat if it is empty, and in a random empty seat if not. What is the probability that the last passenger gets to sit in her own seat?

7. Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials. (Putnam 2007)

8. If α is an irrational number, $0 < \alpha < 1$, is there an almost surely finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is almost surely finite if with probability 1 it must end in a finite number of moves.) (Putnam 1989)

9. I take a (random) walk on the integer number line: At every step, I flip a fair coin, and if it is heads I go one unit to the right, while if it is tails I go one unit to the left. What is the probability that I eventually return to where I started? (Hint: How is this probability related to the expected number of times I return to my original starting position?)