Putnam Club Week Six – 23 October 2012

Combinatorics

1. How many different rectangles with vertices at lattice points, sides parallel to the coordinate axes, and positive area are contained in the rectangle with vertices at $(0,0)$, $(m,0)$, $(0,n)$, and (m,n)?

2. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1,2,\ldots,n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish. (Putnam 1996)

3. Show that the Fibonacci numbers, defined by $f_0 = 1$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$, satisfy the equation $f_{2n} = f_n^2 + f_{n-1}^2$. Come up with as many different proofs as you can.

4. How many permutations of n objects are there that leave no object in its original place?

5. A ternary string (that is, a string of 0s, 1s, and 2s) is *troublesome* if no three consecutive elements are all different. For instance, there are 3 troublesome strings of length 1, 9 of length 2, and 21 of length 3. How many troublesome strings of length 10 are there?

6. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

(Putnam 1999)

7. Write the integers from 1 to n in a circle, so that their distances wrap around (e.g., if $n = 10$, then the number 1 and 9 have distance 2 between them, rather than distance 8 as they would if you wrote them in a line, while 3 and 4 are still distance 1 apart). Call a subset $S \subseteq \{1,2,\ldots,n\}$ *crabby* if every pair of elements of S are distance at least 2 apart. How many crabby subsets of $\{1,2,\ldots,n\}$ are there?

8. Let $\mathcal{P}_n$ be the set of subsets of $\{1,2,\ldots,n\}$. Let $c(n,m)$ be the number of functions $f : \mathcal{P}_n \to \{1,2,\ldots,m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n.$$