Number Theory

1. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

(Putnam 2009)

2. Show that if p and $p^2 + 2$ are primes, then $p^3 + 2$ is also a prime.

3. The integers 2^n and 5^n $(n > 0)$ start with the same digit. What is that digit?

4. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1? (Putnam 1989)

5. Show that there are infinitely many different positive integers which are not the sum of a square and a prime.

6. Prove that for any positive integers a, b, c, we have

$$\frac{[a, b, c]^2}{[a, b][b, c][a, c]} = \frac{(a, b, c)^2}{(a, b)(b, c)(a, c)},$$

where ( ) denotes the greatest common divisor and [ ] the least common multiple. (USAMO 1972)

7. Find, with proof, all possible integer values of $\dfrac{a^2 + ab + b^2}{ab - 1}$, where a and b are positive integers.

8. Show that there are no integers x and y such that $y^2 = x^3 + 7$.