Putnam Club Week Two – 25 September 2012

Polynomials and Algebra

1. Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$P(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of such integers m_1, m_2, m_3, m_4, m_5 for which this minimum is achieved. (Putnam 1985)

2. Find all polynomials $f(x)$ such that $xf(x-1) = (x+1)f(x)$.

3. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

4. Prove that if $x^3 + px^2 + qx + r = 0$ has three real solutions, then $p^2 \geq 3q$.

5. For which real numbers c is there a straight line that intersects the curve

$$x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points? (Putnam 1994)

6. Let $f(x)$ be a monic polynomial with integer coefficients, and let $k \in \mathbb{Z}$, $p \in \mathbb{N}$. Prove that if none of the numbers $f(k), f(k+1), \ldots, f(k+p)$ is divisible by p, then $f(x) = 0$ has no rational solution.

7. Curves A, B, C and D are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\}, \\ B &= \left\{(x,y) : 2xy + \frac{y}{x^2 + y^2} = 3\right\}, \\ C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \\ D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$. (Putnam 1987)

8. Let $p(x)$ be a polynomial with integer coefficients. Suppose that there are three different integers a, b, c such that $p(a) = p(b) = p(c) = -1$. Show that $p(x)$ has no integer roots.

9. Let $P(x)$ ba a monic polynomial with integer coefficients. Suppose that there are four different integers a, b, c, d such that $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k such that $P(k) = 8$. (Canada MO 1970)

10. Let $P(x)$ be a polynomial of degree n, not necessarily with integer coefficients. For how many consecutive integers must $P(x)$ be an integer in order to guarantee that $P(x)$ is an integer for each integer x?