Putnam Club Week One – 18 September 2012

Some Basic Techniques

Extremes, Parity, Divisibility, Coloring, Invariants, Monovariants, and Pigeonhole

1. Show that if an $a \times b$ rectangle can be tiled with $1 \times n$ rectangles, then either $n|a$ or $n|b$.

2. 41 chess rooks are placed on a 10×10 chessboard. Show that there are some five of them, such that none of the five attacks any other.

3. An infinite chessboard contains a positive integer in each square. These have the property that the integer in any square is the average of the integers in its four neighbors. Prove that the integers are all equal.

4. n red points and n blue points, no three collinear, lie in the plane. Show that one can draw n line segments pairing up every point with one of the other color, so that the line segments do not intersect.

5. Let n be an odd integer and $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$. Show that $\prod_{i=1}^{n}(a_i - i)$ is even.

6. 19 squares in a 20×20 chessboard are colored blue at time zero. At each time step, any square with at least two blue neighbors (squares that share an edge with it) also becomes colored blue. Is it possible that eventually the entire board is colored blue?

7. Recall that the Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n > 1$. Suppose that F_n is a prime number. Show that $n = 4$ or n is prime.

8. A rectangle is covered exactly with nonoverlapping smaller rectangles having edges parallel to the edges of the large rectangle, each with at least one edge of integer length. Show that the large rectangle has at least one edge of integer length.

9. For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, and c_i, and area A_i. Suppose that $a_1 \le a_2$, $b_1 \le b_2$, and $c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$? (Putnam 2004)

10. Some nonnegative real numbers are written on the board. At any time, it is permitted to erase two numbers, $a \ge b$, and replace them with $a - b$ and $\sqrt{2ab}$. Suppose that initially the numbers on the board are the integers 0 through 10. Is it possible for the board to ever have a number 20 or greater?