Complex numbers.
11/29/17

(Source: Northwestern University Putnam preparation materials. Some of these problems can actually be solved without complex numbers.)

1. (warm-up) If the integers m and n are expressible as sums of two perfect squares, then so is mn.

2. Show that if z is a complex number such that $z + \frac{1}{z} = 2\cos(\alpha)$, then $z^n + \frac{1}{z^n} = 2\cos(n\alpha)$ for all n.

3. Factor $p(z) = z^5 + z + 1$.

4. Find a closed-form expression for $\sum_{k=0}^{n} \sin(k)$.

5. Find a closed-form expression for $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$.

6. A regular n-gon is inscribed in a unit circle. What is the average length of its diagonal (let us agree that sides of the polygon are considered diagonals, too)? What is the limit of this expression as n goes to infinity? (And just out of curiosity: can you find a closed expression for the geometric mean of the lengths of the diagonals?)

7. (Putnam 1991, B2): Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. You are told that

$$f(x+y) = f(x)f(y) - g(x)g(y)$$
$$g(x+y) = f(x)g(y) + g(x)f(y),$$

and that $f'(0) = 0$. Show that $f(x)^2 + g(x)^2 = 1$ for all x.

8. n lights are arranged in a circle, with exactly one initially on. You are permitted to do the following: given any divisor d of n that is strictly less that n, consider the n/d lights arranged at regular intervals (every d-th light) around the circle. If all lights are in the same state, you are allowed to turn them all on (if they are off) or to turn them all off (if they are on). For which values of n is it possible to turn all the lights on by a sequence of such moves?

Things to remember:

- Formulas for product of complex numbers;
- $|z_1 z_2| = |z_1| \cdot |z_2|$;
- Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$ and its relatives:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

Hints:

1. (warm-up) $|a + bi| = \sqrt{a^2 + b^2}$.

2. Show that $z = e^{\pm \alpha i}$.

3. Look for roots of p among the roots of unity.

4. Use Euler's identity.

5. There is a trick to finding $\prod(1 - \zeta^i)$ where ζ is a root of unity: treat 1 as a variable (!)

6. See the previous problems.

7. Consider $f(x) + g(x)i$ (or simply take $h(x) = f(x)^2 + g(x)^2$ and figure out what equation it satisfies.

8. Assuming the lights are arranged at regular intervals around the origin, what is the sum of the position vectors of lights that are on?