ASSORTED PROBLEMS

These are mostly from (or inspired by) Cornell University's practice Putnam. Obviously, it is intended to be easier than the actual exam, so that it does not take this much time.

Reminder! The Putnam exam is this Saturday! (9-12 and 2-5 in VV B135).

1. Let $p(x)$ be a degree n polynomial (with real coefficients) that has n distinct (real) roots. What is the smallest possible number of non-zero coefficients that $p(x)$ may have?

2. Fix a number α such that $0 < \alpha < 1$. Show that every positive number x can be approximated arbitrarily closely by numbers of the form $n^\alpha - m^\alpha$ for positive integers n and m. (That is, for any $x > 0$ and any $\epsilon > 0$, there exist n and m such that $|x - (n^\alpha - m^\alpha)| < \epsilon$.

3. Let S be a finite set and let $*$ be a binary operation on S such that

$$x * (y * x) = y$$

for any x and $y \in S$. Prove that for any $a, b \in S$, each of the equations $a * x = b$ and $x * a = b$ has a unique solution $x \in S$.

4. Prove that for any prime $p > 17$, the number $p^{32} - 1$ is divisible by 16320.

5. For any two positive numbers a and b, find the sidelength of the smallest cube that contains two non-overlapping spheres of radii a and b. (Here non-overlapping means that their interiors do not intersect.)

6. For each permutation $a_1, \ldots, a_{10}$ of numbers $1, \ldots, 10$, form the sum

$$|a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_9 - a_{10}| + |a_{10} - a_1|.$$

Find the average value of all such sums.