Functions and calculus.
11/01/17

1. Find all polynomials $p(x)$ satisfying
$$p(x+1) = p(x) + 2x + 1.$$

2. Find all functions f with the property that
$$f(x) = f(x/2)$$
for all $x \in \mathbb{R}$.

3. (VT 2007, #2). Given that
$$e^x = 1/0! + x/1! + x^2/2! + \cdots + x^n/n! + \ldots,$$
find, in terms of e, the exact values of
$$1/1! + 2/3! + 3/5! + \cdots + n/(2n-1)! + \ldots$$
and
$$1/3! + 2/5! + 3/7! + \cdots + n/(2n+1)! + \ldots.$$

4. (VT 2008, #1). Find the maximum value of
$$xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x$$
where $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

5. (VT 2011, #7). Let
$$P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1x + 1$$
be a polynomial where the a_i ($1 \leq i \leq 97$) are real numbers. Prove that the equation $P(x) = 0$ has at least one complex root (i.e., a root of the form $a+bi$ with a, b real numbers and $b \neq 0$).

6. (Putnam 2009, A1). Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane,
$$f(A) + f(B) + f(C) + f(D) = 0.$$
Does it follow that $f(P) = 0$ for all points P in the plane?

7. (Putnam 2010, A2). Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f'(x) = \frac{f(x+n) - f(x)}{n}$$
for all real numbers x and all positive integers n.

8. (Putnam 2008, B5). Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with
$$\gcd(a, b) = 1.)$$
(Note: gcd means greatest common divisor.)

Hints:

1. What is $\deg(p(x+1) - p(x))$?

2. Once you are allowed to multiply or divide x by 2, is there some 'normal form' that you can bring it to?

3. What is the relation between the Taylor power series of $f(x)$ and that of $f(x)+f(-x)$? that of $f'(x)$?

4. Note the symmetry of the expression, and try maximizing it one variable at a time.

5. Use the fact that the derivative must vanish at least once between the roots of the polynomial.

6. The trick is to combine several instances of the relation in a non-trivial way.

7. What can you say about the derivative of f?

8. Use the linear approximation of f near a rational point (say, 0) to determine f's derivative at that point.