Polynomials.
10/23/13

1. What is the remainder when the polynomial $x^{50} + 50$ is divided by $x - 1$?

2. What is the remainder when the polynomial $x^{50} + 50$ is divided by $x^2 - x$?

3. (VT 1992, #6) Let $p(x)$ be the polynomial

$$p(x) = x^3 + ax^2 + bx + c.$$

Show that if $p(r) = 0$, then

$$\frac{p(x)}{x-r} - \frac{2p(x+1)}{x+1-r} + \frac{p(x+2)}{x+2-r} = 2$$

for all x except $x = r$, $r - 1$ and $r - 2$.

4. (VT 1995, #3) Let $n \geq 2$ be a positive integer and let $f(x)$ be the polynomial

$$1 - (x + x^2 + \cdots + x^n) + (x + x^2 + \cdots + x^n)^2 - \cdots + (-1)^n (x + x^2 + \cdots + x^n)^n.$$

If r is an integer such that $2 \leq r \leq n$, show that the coefficient of x^r in $f(x)$ is zero.

5. (Putnam 2007, B1) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

6. Let $f(x)$ be a polynomial with integral coefficients, and $k > 1$ is an integer. Suppose none of the numbers $f(1), f(2), \ldots, f(k)$ are divisible by k. Prove that $f(x)$ has no integral roots.

7. (Putnam 2005, B1) Find a nonzero polynomial $P(x, y)$ such that $P([a], [2a]) = 0$ for all real numbers a. (Note: $[v]$ is the greatest integer less or equal to v.)