VIRGINIA TECH PRACTICE (10/21/15)

1. (Inspired by VT'10 problem) Let d be a positive integer and let A be a $d \times d$ matrix with integer entries. Suppose

$$I + A + A^2 + \cdots + A^{100} = 0,$$

where I is the identity matrix.

(a) Prove that $\det(I + A) = \pm 1$.

(b) Determine for which k between 0 and 100 we have $\det(I + A + \cdots + A^k) = \pm 1$.

2. (VT'94) Evaluate

$$\int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2} dz dy dx.$$

3. (VT'10) For n a positive integer, define $f_1(n) = n$ and then for i a positive integer, define

$$f_{i+1}(n) = f_i(n)^{f_i(n)}.$$

Determine $f_{100}(75) \mod 17$ (i.e. determine the remainder after dividing $f_{100}(75)$ by 17, an integer between 0 and 16). Justify your answer.

4. (VT'09)A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the n-th time (so $f(0) = 0$). Find a formula for $f(n)$.

5. We wish to tile a strip of n 1-inch by 1-inch squares. We can use dominos which are made up of two tiles which cover two adjacent squares, or 1-inch square tiles which cover one square. We may cover each square with one or two tiles and a tile can be above or below a domino on a square, but no part of a domino can be placed on any part of a different domino. We do not distinguish whether a domino is above or below a tile on a given square. Let $t(n)$ denote the number of ways the strip can be tiled according to the above rules. Thus for example, $t(1) = 2$ and $t(2) = 8$. Find a recurrence relation for $t(n)$, and use it to compute $t(6)$.