Number theory.
10/09/13

Number theory is a common topic.

1. Among five integers, there are always three with sum divisible by 3.

2. Among $n+1$ positive integers no greater than $2n$, there are two that are coprime.

3. If p and p^2+2 are primes, then p^3+2 is also a prime.

4. (VT 2006, #3) Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2006)$ (e.g. the last digit of 2006 is 6).

5. (VT 2011, #4). Let m, n be positive integers and let $[a]$ denote the residue class $\mod mn$ of the integer a (thus

$$\{[r] | r \text{ is an integer}\}$$

has exactly mn elements). Suppose the set

$$\{[ar] | r \text{ is an integer}\}$$

has exactly m elements. Prove that there is a positive integer q such that q is prime to mn and $[nq] = [a]$.

6. (VT 2012, #4). Define $f(n)$ for n a positive integer by

$$f(1) = 3 \text{ and } f(n+1) = 3^{f(n)}.$$

What are the last two digits of $f(2012)$?

7. (Putnam 2003, B3). Show that

$$\prod_{i=1}^{n} \operatorname{lcm}(1, 2, 3, \ldots, [n/i]) = n!$$

8. (Putnam 2000, A6). $p(x)$ is a polynomial with integer coefficients. A sequence $x_0, x_1, x_2, \ldots$ is defined by

$$x_0 = 0, \quad x_{n+1} = p(x_n).$$

Prove that if $x_n = 0$ for some $n > 0$, then $x_1 = 0$ or $x_2 = 0$.