POLYNOMIALS (10/06/21)

Easier problems

1. The polynomial $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ has integer coefficients and n distinct integer roots. Suppose that all of its roots are coprime. Show that a_n and a_{n-1} are coprime.

2. a, b, and c are the three roots of the polynomial $x^3 - 3x^2 + 1$. Find $a^3 + b^3 + c^3$.

3. $p(x)$ is a polynomial with integer coefficients such that $p(0) = 1$, $p(1) = 2$, $p(-1) = 4$. Prove that $p(x)$ has no integer roots. (In the first version of this problem, I had $p(-1) = 4$. Why did I changed it?)

4. (AIME 1989) Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

Putnam problems (not necessarily harder)

5. (Putnam 2007) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

6. (Putnam 2009) Let $p(x)$ be a real polynomial that is nonnegative for all real x. Prove that for some k, there are real polynomials $f_1(x), \ldots, f_k(x)$ such that $p(x) = \sum_{i=1}^{k} f_i(x)^2$.

7. (Putnam 2003) Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)?$$

8. (Putnam 2010) Find all polynomials $P(x)$, $Q(x)$ with real coefficients such that $P(x)Q(x + 1) - P(x + 1)Q(x) = 1$.

9. (Putnam 2008) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points

$$(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n)) \in \mathbb{R}^2$$

are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater or equal to $n - 1$. (By the way, suppose n is odd and $\deg(f(x)) < n - 1$. Can you say anything special about the regular n-gon?)

10. (Putnam 2005) (This is quite hard.) Let $p(z)$ be a polynomial of degree n all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z)$ have absolute value 1.

Good things to know

- The Fundamental Theorem of Algebra.
- Expressions for coefficients in terms of roots (as elementary symmetric polynomials).
- Roots of real polynomials (must occur in conjugate pairs).
- Intermediate Value Theorem.
- Mean Value Theorem: For a real-valued function, the derivative must vanish between two zeros of the function.
- Divisibility: for an integer polynomial $p(x)$, $(a-b)|(p(a)-p(b))$.

Hints

1. Think about how a_i's are expressed in terms of roots.

2. Express $a^3+b^3+c^3$ in terms of the elementary symmetric polynomials.

3. What is $p(x)$ modulo 3?

4. The left-hand sides are values of a (low-degree!) polynomial.

5. Not much to say - straight divisibility question.

6. What does the factorization of $p(x)$ look like?

7. Consider the coefficients of y^k on the right-hand side as polynomials of x. How are they related to $a(x)$ and $b(x)$? (Or plug in particular values of y into both sides.)

8. Compare $P(x)(Q(x+1)-Q(x))$ and $Q(x)(P(x+1)-P(x))$.

9. What can you say about the vector-valued polynomial function

$$(f(x+1)-f(x), g(x+1)-g(x))$$

?

10. What is special about the values of $g(z)$ if $|z|=1$? Also, think about the Mean Value Theorem.