Fall 2017

Previous math competitions problems-Putnam & Virginia Tech

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1. Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

2. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

3. Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

4. Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does
$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$
converge?

5. Show that, for all integers $n > 1$,
$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

6. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions
$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1. \end{aligned}$$
Find an explicit formula for $f(x)$, valid in some open interval around 0.

7. Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y)\,dy$, $b = \int_0^1 f(1,y)\,dy$, $c = \int_0^1 f(x,0)\,dx$, $d = \int_0^1 f(x,1)\,dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0,1)^2$ such that
$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.$$

8. Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

9. Let $c > 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : R \to R$ such that $f(x) = f(x^2 + c)$ for all $x \in R$. Note that R denotes the set of real numbers.

10. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

11. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1$, $f(4) = 2$ and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that

$$f(n) = n^2 - 4n - 18.$$

Determine all positive values for n.

12. Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\left(2^{n+1} - 1\right).$$