9/30 – The Pigeonhole Principle

(1) Seven points are chosen in a regular hexagon of side length 1 unit. Show that some two of them are at most 1 unit apart.

(2) Let $x_1, \ldots, x_6$ be any integers. Prove that $\prod_{1 \leq m < n \leq 6} (x_n - x_m)$ is divisible by 5!.

(3) 10 guests sit down at a round table at a dinner party. They notice that none of them is sitting at his or her assigned seat. Show that there is some rotation of their seating arrangement such that at least two people are in their assigned seats.

(4) Show that of any $n+1$ integers between 1 and $2n$ (inclusive), some pair of them are relatively prime.

(5) Show that of any $n+1$ integers between 1 and $2n$ (inclusive), there are two such that one divides the other.

(6) Fifteen positive integers are chosen less than 100. Show that there must be two distinct pairs of these integers whose differences are the same. (That is, there are $a < b \leq c < d$ in the set such that $b - a = d - c$.)

(7) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere. (Putnam 2002)

(8) 14 positive integers are chosen less than 1000. Show that there is some pair of disjoint subsets of these numbers whose sums are equal.

A problem for you to think about at home:

Let f_n be the Fibonacci sequence, which is given by $f_1 = f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

(a) Prove that the sequence of last digits of f_n is periodic.

(b) Prove that there is some n such that f_n is divisible by 2015.