Fall 2017

# Equations with functions as unknowns

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1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 - y^2) = (x - y)\,(f(x) + f(y))\,.$$

2. Find all functions $f : \mathbb{R} - \{1\} \to \mathbb{R}$, continuous at 0, that satisfy

$$f(x) = f\left(\frac{x}{1-x}\right), \quad x \in \mathbb{R} - \{1\}\,.$$

3. Find all functions $f : [0, 1] \to \mathbb{R}$ satisfying the following conditions

   - $[f(x)] \sin^2 x = [x] \cos f(x) \cos x = f(x)$
   - $f(f(x)) = f(x)$.

   Here $[x]$ means the fractional part of $x$.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals $x$, $y$, $z$, we have

$$[f(x) + 1]\,[f(y) + f(z)] = f(xy + z) + f(xz - y).$$

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\,(f(x) + y) = f\,(f(x) - y) + 4f(x)y, \text{ for any } x, y \in \mathbb{R}.$$

6. Let $c$ be a positive integer. The sequence $a_1, a_2, \ldots$ is defined by

$$a_1 := c, \text{ and } a_{n+1} = a_n^2 + a_n + c^3, n \in \mathbb{N}.$$

   Find all values of $c$ for which there exist some integers $k \geq 1$ and $m \geq 2$ such that $a_k^2 = c^3$ is the $m$-th power of some positive integer.

7. (Putnam) Find all functions $f$ from the interval $(1, \infty)$ to $(1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$, then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

8. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which thre is a positive real number $a$ such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)},$$

   for all $x > 0$.