Extremes.
03/29/23

A few simpler problems this time. They are loosely based on the 'Extreme Principle': if a quantity is important to a problem, consider its extrema (max or min). (Of course, you might have to show the extrema exist...)

This general problem-solving principle is useful on all levels, and there are several 'timeless classics' on the list (apologies if you have seen them before - I couldn't make myself skip them).

1. Does there exists a tetrahedron such that every edge is a side of an obtuse angle on a face? (That is, on one of the two adjacent faces, one of the two angles is obtuse.)

2. The integers $1, 2, \ldots, n^2$ are placed on the $n \times n$ square board. Prove that there are two adjacent (vertically, horizontally, or diagonally) squares whose numbers differ by at least $n + 1$. Can this bound be improved?

3. All plane sections of a solid are circles. Prove that the solid is a ball.

4. There is a set S of points in the plane with the property that any triangle with vertices in S has area at most 1. Prove that there exists a triangle with area 4 containing all the points of S.

5. Each member of a parliament has at most three enemies. Prove that the parliament can be split into two houses in such a way that each member has no more than one enemy in his house.

6. 2014 vector are drawn on a plane. Two players alternately take a vector until there are no more vectors left. The winner is the player whose vectors' sum is longer. Suggest a winning (or at least a non-losing) strategy for the first player.

7. n identical cars are stopped on a circular track. Together, they have enough gas to complete one lap. Show that one of the cars can complete a lap by collecting fuel from the other cars on the way.

8. Let $p(x)$ be a real polynomial such that for all x, $p(x)+p'(x) \geq 0$. (For example, $p(x) = x^2 + 1$.) Does it follow that for all x, $p(x) \geq 0$?

9. (Putnam 1979) Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red, and the remaining n blue. Prove or disprove: there are n straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.