

Games.
3/22/23

1. Consider the following two-player game. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny which is already on the table. The table starts out with no pennies. The last player who makes a legal move wins. Which player has a winning strategy?
2. There are 2013 boxes containing $1, 2, 3, \dots, 2013$ chips respectively on the table. You may choose any subset of boxes, and subtract the same number of chips from each box. What is the minimal number of moves you need to empty all boxes?
3. Two players alternatively write positive integers not exceeding n on the board. Writing divisors of numbers that are already written is not allowed. Whoever cannot write a number loses. Which player has a winning strategy if (a) $n = 10$ (b) $n = 1000$?
4. Alice crosses out any 2^7 of the numbers $0, 1, \dots, 256$. Next Bob crosses out any 2^6 of the remaining numbers. Then Alice crosses out any 2^5 of the remaining numbers, and so on, until finally Bob crosses out $2^0 = 1$ number. After this, there are only two numbers, a and b left, and Bob pays the difference $|a - b|$ to Alice. How should Alice play to win as much as she can? How should Bob play to lose as little as possible? What is the amount $|a - b|$ if both players play optimally?
5. A and B play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B win the game?
6. (Putnam 2006, A2). Alice and Bob play a game in which they take turns removing stones from a heap that initially had n stones. The number of stones removed each turn must be one less than a prime number. The winner is the player who takes the last stone. Prove that there are infinitely many values of n for which Bob has the winning strategy.
7. (Putnam 2008, A2). Alan and Barbara play a game in which they take turns filling an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when the entire array is filled. Alan wins if the determinant of the resulting matrix is non-zero; Barbara wins if it is zero. Which player has a winning strategy?
8. (Putnam 1995, B5). The game starts with four heaps of beans, containing 3, 4, 5, and 6 beans. The two players move alternately. A move consists of taking either
 - (a) One bean from a heap, provided at least two beans are left behind in that heap; or
 - (b) A complete heap of two or three beans.The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.