FUNCTIONAL EQUATIONS (02/15/23)

1. Warm-up: Standard (but useful) problems

Unless stated otherwise, 'functions' means functions of one (real) variable.

1. What functions f satisfy $f(2x) = f(x)$? What changes if we ask that f must be continuous?

2. What functions f satisfy $f(x+y) = f(x) + f(y)$? What changes if we require that f must be continuous?

3. What functions f satisfy $f(x) + xf(-x) = x - x^2$?

2. Some suggestions

(1) Does the equation imply something special for special values of the variable? For instance, what happens if the variable is 0? 1? Maybe there is some other value that makes the equation simpler?
(2) If the function is supposed to be continuous, consider what happens as we take the limit (maybe when approaching some special value). If it is supposed to be differentiable, can we say something about its derivative?
(3) Look for a way to combine the equation with itself...

3. More problems of the same type

4. What continuous functions $f(x)$ satisfy $f(2x+1) = f(x)$? What about the equation $f(2x+1) = f(x) + 1$? Or $f(2x+1) = 2f(x)$?

5. What functions satisfy $f(xy) = f(x) + f(y)$? What changes if f is required to be continuous?

(Note that both this and the previous equation are 'linear'... Does this help?)

6. What functions defined for $x > 0$ satisfy $f(x)f(1/x)^2 = x^3$?

4. Slightly different problems

(We probably won't have time to look at all of these, so they are more of a homework.)

7. (Putnam'08) Let f be a function of two variables satisfying the equation

$$f(x,y) + f(y,z) + f(z,x) = 0$$

for all x, y, z. Show that there exists a single-variable function g such that $f(x,y) = g(x) - g(y)$. (The statement is actually deeper than it seems: it can be interpreted as a calculation of some *cohomology*.)

8. Find all functions $f(x)$ defined for $x \geq 0$ that satisfy

$$f((x+y)^2) = f(x^2+y^2).$$

9. Find all continuous functions $f(x)$ defined for $x \geq 0$ such that $f(x^2+y^2) = f(x)^2 + f(y)^2$.

10. (Putnam'00) Let $f(x)$ be a continuous function such that $f(2x^2-1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for all $-1 \leq x \leq 1$.

11. Find all functions $f(x)$ such that $f(\frac{x+y}{2}) + f(x) + f(y) = x + y$.

12. $f(x,y)$ is a function on the plane. It has the following property: the sum of its values in the vertices of any square is equal to zero. Show that $f = 0$.