Putnam Club Problem Sheet - February 1

1. Concavity

Warmup Problem Show that for any ΔABC, $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.

Warmup Problem If $a_1, \ldots, a_n$ are nonnegative, then

$$(1 + a_1) \cdot \ldots \cdot (1 + a_n) \geq \left(1 + (a_1 \cdot \ldots \cdot a_n)^{1/n}\right)^n$$

Problem. (2001 IMO Problem 2) Show that if a, b and c are positive real numbers, then

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ac}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

Hint: If the inequality holds for (a, b, c) then it holds for $(\lambda a, \lambda b, \lambda c)$ for any $\lambda > 0$. Assume wlog that $a + b + c = 1$ and hence that the LHS is a weighted average.

2. AM-GM Inequality

Problem. (2021 B2) Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying $\sum_{k=1}^{\infty} a_k = 1$. **Hint:** Notice that

$$\frac{1}{2^n} = \frac{1}{2}\left(\frac{1}{2^{n(n-1)}}\right)^{1/n} = \frac{1}{2}\left(\frac{1}{4^{\frac{n(n-1)}{2}}}\right)^{1/n} = \frac{1}{2}\left(\prod_{k=1}^{n} \frac{1}{4^{n-k}}\right)^{1/n}$$

Problem. (2019 A3)

Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \ldots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

3. Hölder's Inequality and Cauchy-Schwarz

Warmup Problem. Fix real numbers $a < b$. Let $\mathcal{C}$ be the set of continuous functions $f : [a,b] \longrightarrow (0,\infty)$. Find $\max_{f\in\mathcal{C}} \left(\frac{\left(\int_a^b f\right)^3}{\int_a^b f^3} \right)$.

Problem. (2010 B1) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Problem. (2012 B2 - variant) Prove that there is a constant $c > 0$ with the following property. Let F be a face of area A of a polyhedron in $\mathbb{R}^3$. If a collection of n balls whose volumes sum to V contains F, then $nV^2 > cA^3$.