Spring 2021

Problem set - Linear Algebra and some Analysis

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Various problems from Linear Algebra and Analysis

Linear Algebra

1. Consider given the following matrix $A = I_n - 2XX^T$, where $X = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, and I_n is the n-dimensional identity matrix . In addition we know that the entries of the vector X satisfy the following two contains: $x_1 + x_2 + \cdots + x_n \neq 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$.
   - Prove that matrix A is an orthogonal matrix.
   - Find the eigenvalues of A and a basis with respect to which matrix A is expressed canonically (Jordan form).
   - Find X knowing that $(1, 1, 1, \dots, 1)^T$ is a eigenvector of A.
2. Prove that for any nonzero square matrix A there exists a matrix X such that the matrices X and $A + X$ have no common eigenvalues.
3. Let $A \in \mathcal{M}_n(\mathbb{Q})$ such that $det(A + \sqrt[n]{2I_n}) = 0$. Prove that
$$det(A - I_n) = detA + (det(A + I_n))^n.$$
4. Let $n \in \mathbb{N}, n \geq 2$. For any square matrix $A \in \mathcal{M}_n(\mathbb{C})$, we denote by $m(A)$ the number of all its nonzero minors . Prove that
   - $m(I_n) = 2^n - 1$
   - If $A \in \mathcal{M}_n(\mathbb{C})$ is nonsingular then prove that $m(A) \geq 2^n - 1$.
5. Let $P(t)$ be a polynomial of even degree with real coefficients. Prove that the function $f(x) = P(x)$ defined in the set of $n \times n$ matrices is not onto.
6. Let n be an odd positive integer and A an $n \times n$ matrix with the property that $A^2 = 0_n$ or $A^2 = I_n$. Prove that
$$det(A + I_n) \geq det(A - I_n).$$
7. Let A be an $n \times n$ matrix such that there exists a positive integer k for which
$$kA^{k+1} = (k + 1)A^k.$$
Prove that the matrix $A - I_n$ is invertible and find its inverse.

8. Let A and B be $n \times n$ matrices for which there exists $a, b \in \mathbb{R} \setminus \{0\}$ such that $AB = aA + bB$. Prove that A and B commute.

9. Let V be a vector space over $\mathbb{R}$, and let's assume it is of finite dimension. Let $T, S : V \to V$ be to linear maps that satisfy the properties $T \circ S = T$ and $S \circ T = S$. Prove that

   - rang T =rang S (rang = the dimension of the image of the linear map).
   - if Im T=Im S then $S = T$.

10. Let U and V be isometric linear transformations of $\mathbb{R}^n$, $n \geq 1$, with the property that

$$\|Ux - x\| \leq \frac{1}{2}, \text{ and } \|Vx - x\| \leq \frac{1}{2}, \forall x \in \mathbb{R}^n \text{ with } \|x\| = 1.$$

Prove that

$$\|UVU^{-1}V^{-1}x - x\| \leq \frac{1}{2} \forall x \in \mathbb{R}^n \text{ with } \|x\| = 1.$$

Analysis

1. We define the following sequence of real numbers, $\{x_n\}_{n\geq 1}$, as follows

$$x_n = \sum_{1 \leq i < j \leq n} \frac{1}{i \cdot j}, \quad \forall n \geq 1.$$

What can you say about the following series

$$\sum_{n=1}^{\infty} \frac{1}{x_n^{\alpha}},$$

where α is a real number ?

2. Let $n \in \mathbb{N}^*$. Decide if the following limit exists:

$$\lim_{(x_1, x_2, \ldots, x_n) \to (0, 0, \ldots, 0)} \frac{x_1 x_2 \cdots x_n}{x_1^1 + x_2^2 + \cdots + x_n^2}.$$

Can you find the limit?