Fall 2020

Problem set 1

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Theoretical Introduction to Generating functions

Here are basic recurrence equations that you can solve:

1. Let a_n be a sequence given by $a_0 = 0$ and $a_{n+1} = 2a_n + 1$ for $n \geq 1$. Find the general term of the sequence a_n.

2. Find the general term of the sequence given recurrently by

$$a_{n+1} = 2a_n + n, \quad (n \geq 0), \quad a_0 = 1.$$

3. $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. Find the general term of the sequence.

4. Let the sequence be given by $a_0 = 0$, $a_1 = 2$, and for $n \leq 0$:

$$a_{n+2} = -4a_{n+1} - 8a_n.$$

   Find the general term of the sequence.

5. Find the general term of the sequence x_n given by

$$x_0 = x_1 = 0, \qquad x_{n+2} - 6x_{n+1} + 9x_n = 2^n + n \quad \text{for } n \geq 0.$$

6. Let $f_1 = 1$, $f_{2n} = f_n$, and $f_{2n+1} = f_n + f_{n+1}$. Find the general term of the sequence.

7. Evaluate the sum

$$\sum_k \binom{k}{n-k}.$$

8. Evaluate the sum

$$\sum_{k=m}^{n} (-1)^k \binom{n}{k} \binom{k}{m}.$$

9. Evaluate the sum

$$\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m}$$

10. Evaluate

$$\sum_k \binom{n}{\left[\frac{k}{2}\right]} x^k.$$

11. Determine the elements of the sequence:

$$f(m) = \sum_k \binom{n}{k} \binom{n-k}{\left[\frac{m-k}{2}\right]} y^k.$$

12. Prove that

$$\sum_{k=0}^{n} \binom{2n}{2k} \binom{2k}{k} 2^{2n-2k} = \binom{4n}{2n}.$$

The following problem is slightly harder because the standard idea of snake oil doesnt lead to a solution.

13. For given n and p evaluate

$$\sum_k \binom{2n+1}{2p+2k+1} \binom{p+k}{k}.$$

14. Prove that for the sequence of Fibonacci numbers we have

$$F_0 + F_1 + \cdots + F_n = F_{n+2} + 1.$$

15. Given a positive integer n, let A denote the number of ways in which n can be partitioned as a sum of odd integers. Let B be the number of ways in which n can be partitioned as a sum of different integers. Prove that $A = B$.

3. Find the number of permutations without fixed points of the set

$$\{1, 2, \ldots, n\}$$

16. Let $n \in \mathbb{N}$ and assume that

$$x + 2y = n \quad \text{has } R_1 \text{ solutions in } \mathbb{N}_0^2$$

$$2x + 3y = n - 1 \quad \text{has } R_2 \text{ solutions in } \mathbb{N}_0^2 \vdots$$

$$nx + (n+1)y = 1 \quad \text{has } R_n \text{ solutions in } \mathbb{N}_0^2$$

$$(n+1)x + (n+2)y = 0 \quad \text{has } R_{n+1} \text{ solutions in } \mathbb{N}_0^2$$

Prove $\sum_k R_k = n + 1$.

17. A polynomial $f(x_1, x_2, \dots x_n)$ is called a *symmetric* if each permutation $\sigma \in S_n$ we have $f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$. We will consider several classes of symmetric polynomials. The first class consists of the polynomials of the form:

$$\sigma_k(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_k} x_{i_1} \cdot \dots \cdot x_{i_k}$$

for $1 \leq k \leq n$, $\sigma_0 = 1$, $\sigma_k = 0$ for $k > n$. Another class of symmetric polynomials are the polynomials of the form

$$p_k(x_1, \dots, x_n) = \sum_{i_1 + \dots + i_n = k} x_1^{i_1} \cdot \dots \cdot x_n^{i_n}, \qquad \text{where } i_1, \cdots, i_n \in \mathbb{N}_0.$$

The third class consists of the polynomials of the form:

$$s_k(x_1, \dots, x_n) = x_1^k + \dots + x_n^k.$$

Prove the following relations between the polynomials introduced above:

$$\sum_{r=0}^{n} (-1)^r \sigma_r p_{n-r} = 0, \; np_n = \sum_{r=1}^{n} s_r p_{n-r}\,, \text{ and } n\sigma_n = \sum_{r=1}^{n} (-1)^{r-1} s_r \sigma_{n-r}.$$

18. Prove that there is a unique way to partition the set of natural numbers in two sets ?? and B such that: For very non-negative integer n (including 0) the number of ways in which n can be written as $a_1 + a_2, a_1, a_2 \in A$, $a_1 \neq a_2$ is at least 1 and is equal to the number of ways in which it can be represented as $b_1 + b_2$, $b_1, b_2 \in B$, $b_1 \neq b_2$.

19. Prove that in the contemporary calendar the 13th in a month is most likely to be Friday. Remark: The contemporary calendar has a period of 400 years. Every fourth year has 366 days except those divisible by 100 and not by 400.

20. Let a and b be positive integers. For a nonnegative integer n let $s(n)$ be the number of nonnegative integer solutions to the equation

$$ax + by = n$$

Prove that the generating function of the sequence($s(n))_n$ is

$$f(x) = \frac{1}{(1 - x^a)(1 - x^b)}$$

21. Prove that the number of ways of writing n as a sum of distinct positive integers is equal to the number of ways of writing n as a sum of odd positive integers.Note: This property is usually phrased as follows: Prove that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Generating functions are powerful tools for solving a number of problems mostly in combinatorics, but can be useful in other branches of mathematics as well.