Putnam Club Problem Sheet - December 3

Warmup: Geometric configurations.

Putnam 2011 A1. Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0,0), P_1, \ldots, P_n$ such that $n \geq 2$ and:

- the directed line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$ are in the successive coordinate directions east (for P_0P_1), north, west, south, east, etc.;
- the lengths of these line segments are positive and strictly increasing.

How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n of any growing spiral?

Putnam 2021 A1. A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Putnam 2017 B1. Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.

Triangles.

Putnam 2012 A1. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Putnam 2015 A1. Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

Putnam 2019 A2. In the triangle $\triangle ABC$, let G be the centroid (i.e. center of mass), and let I be the center of the inscribed circle (i.e. largest circle contained in the triangle). Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α. (Hint: Recall that the center of the inscribed circle is the intersection of the three angle bisectors).

Putnam 2010 B2 (Variant). What is the smallest possible side length of a nondegenerate triangle in the plane whose vertices have integer coordinates and all of whose sides have integer lengths. .

Putnam 2016 B3. Suppose that S is a finite set of points in the plane such that the area of triangle $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.

Putnam 2017 B5. A line in the plane of a triangle T is called an *equalizer* if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

Putnam 2018 A6. Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle ABD)}$$

is a rational number.

Icosahedra.

Putnam 2013 A1. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Putnam 2017 A6. The 30 edges of a regular icosahedron are distinguished by labeling them $1, 2, \ldots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color? (Hint: Label each color by an element of $\mathbb{Z}/3$ and then interpret the condition that two edges are the same color and one is different. Finally note that 5 edges meet at each vertex of the icosahedron.)