POLYNOMIALS

1. a, b, and c are the three roots of the polynomial $x^3 - 3x^2 + 1$. Find $a^3 + b^3 + c^3$.

2. (AIME 1989) Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

3. (1999-A1, leftover from last time) Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}.$$

4. (2007-B1) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

5. (2016-A1) Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left. \frac{d^j p(x)}{dx^j} \right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

6. (1999-A2) Show that for some fixed positive n we can express every polynomial with real coefficients which is nowhere negative as a sum of the squares of n polynomials.

Harder

7. (2007-B4) Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(x))^2 + (Q(x))^2 = X^{2n} + 1$$

and $degP > degQ$.

8. (2007-B5). Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ such that, for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

Good things to know

- The Fundamental Theorem of Algebra.
- Expressions for coefficients in terms of roots (as elementary symmetric polynomials).
- Roots of real polynomials (must occur in conjugate pairs).
- Intermediate Value Theorem.
- Mean Value Theorem: For a real-valued function, the derivative must vanish between two zeros of the function.
- Divisibility: for an integer polynomial $p(x)$, $(a-b)|(p(a)-p(b))$.

Hints

1. Express $a^3+b^3+c^3$ in terms of $(a+b+c)$, $(ab+ac+bc)$, and abc.

2. The left-hand sides are values of a (low-degree!) polynomial.

3. At each point where the formula for the function changes, what is the difference between the formulas on the two sides?

4. Compare $f(f(n)+1)$ and $f(1)$.

5. $2016 = 2^5 \times 3^2 \times 7$.

6. What does the factorization of such a polynomial look like?

7. This is related to the previous problem. Write $P^2+Q^2 = (P+iQ)(P-iQ)$.

8. Consider the difference $\left\{\frac{n}{k}\right\} = \frac{n}{k} - \left\lfloor\frac{n}{k}\right\rfloor$; it only takes k distinct values. Rewrite both sides in terms of n and $\left\{\frac{n}{k}\right\}$.