

ASSORTED COMPETITION PROBLEMS

WARM-UP

(Based on *Problem-Solving Strategies* by A. Engel).

1. Prove that the equation

$$2^x = [\sqrt{2y}]$$

has infinitely many integral solutions.

2. In space, there are 2023 spherical planets of radius one. On the surface of each planet, we mark the points from which at least one other planet is visible. Find the total marked area.

COMPETITION PROBLEMS: VIRGINIA TECH REGIONAL MATH CONTEST

3. (2022, #4) Calculate the exact value of the series

$$\sum_{n=2}^{\infty} (\log(n^3 + 1) - \log(n^3 - 1)).$$

4. (2003, #3) Find all complex 2×2 matrices A such that $A = A^{-1} = A^t$, where A^t is the transpose of A .

COMPETITION PROBLEMS: UW UNDERGRADUATE MATH COMPETITION

5. (2016, #5) $f(x)$ is a differentiable function satisfying the following conditions:

$$\begin{aligned} 0 < f(x) < 1 & \quad \text{for all } x \text{ on the interval } 0 \leq x \leq 1 \\ 0 < f'(x) < 1 & \quad \text{for all } x \text{ on the interval } 0 \leq x \leq 1. \end{aligned}$$

How many solutions does the equation

$$\underbrace{f(f(f \dots f(x) \dots))}_{2016 \text{ times}} = x$$

have on the interval $0 \leq x \leq 1$?

ACTUAL PUTNAM PROBLEMS

6. (1999-A1) Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x ,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0. \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

7. (2007-A1) Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

8. (2004-B4) Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy -plane, for $k = 1, 2, \dots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k . Let R denote the map obtained by applying, in order, R_1 , then R_2, \dots , then R_n . For an arbitrary point (x, y) , find, and simplify, the coordinates of $R(x, y)$.

UW PUTNAM CLUB INFO

Meeting time: Wednesdays 5–6:30pm, VV B325.

Putnam competition: First Saturday in December (December 2, 2023). Two three-hour sessions of six problems each. Over 2,000 college students participate. This is an individual competition, but the top three scores from an institution count as its ‘institution score’.

UW Undergraduate Math Competition: Spring 2024 (probably April). Easier and shorter (6 problems, 3 hours).

Typical topics: Linear algebra, elementary number theory, calculus, combinatorics; emphasis on problem-solving.