Putnam Club
September 25, 2024
Generating series

A sequence a_n is encoded by its *generating series (or function)*

$$\sum_{n=1}^{\infty} a_n t^n.$$

The problems below can be solved using this idea. Some of them can also be solved without generating functions.

1. (Warmup) Show that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

(This can be done using combinatorics, but, just for fun, how do we use generating functions here?)

2. (Warmup #2) Show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

3. (Warmup #3) A sequence a_n is given by $a_0 = 0$ and $a_{n+1} = 2a_n + 1$. Find the general term of the sequence a_n.

4. (MIT Putnam seminar) Let S_n be the number of triplets of nonnegative integers (a, b, c) such that $a + 2b + 3c = n$. Compute

$$\sum_{n=0}^{\infty} \frac{S_n}{3^n}.$$

5. Show that, for any number n, the number of ways to write n as a sum of distinct positive integers equals the number of ways to write n as a sum of odd integers (not necessarily distinct).

6. (Proofs from the book) The set of natural numbers is partitioned into finitely many arithmetic progression $\{a_i + dr_i\}$, $1 \le i \le n$.

Prove that $\sum_{i=1}^{n} \frac{1}{r_i} = 1$.

7. (Harder) In the situation of the previous problem, show that

$$\sum_{i=1}^{n} \frac{a_i}{r_i} = \frac{n-1}{2}.$$

8. (MIT Putnam seminar) Solve the recurrence Solve the recurrence

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0,$$

with the initial conditions $a_0 = 2, a_1 = 3$.

9. (Putnam'00) Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \ldots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a - 1$ or a is in S_n. Show that there are infinitely many integers N for which $S_N = S_0 \cap \{N + a : a \in S_0\}$.

10. (IMC 2015) Consider all 26^{26} words of length 26 in the Latin alphabet. Define the weight of the word as $\frac{1}{k+1}$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.

Hints:

1. Note that

$$\sum_{n=0}^{\infty} n a_n t^{n-1} = \left(\sum_{n=0}^{\infty} a_n t^n \right)'.$$

2. What happens when we multiply series? (Keep in mind that $\binom{n}{k} = \binom{n}{n-k}$.)

3. Rewrite the recurrence using the generating series. (The problem is easier if you know partial fraction.)

4. Consider the series $(1 + t + t^2 + \dots)$, $(1 + t^2 + t^4 + \dots)$, $(1 + t^3 + t^6 + \dots)$.

5. Not much to say - write the generating series for both expressions. (Do the series actually converge? Also, do we care?)

6. Simplify $\sum_d x^{a_i + dr_i}$.

8. See the first problem.

9. Consider what happens when $S_0 = \{1\}$. (Extra question - how is this a generating series problem?)

10. I would consider the quantity $a(k, n)$, which is the number of words of length n using exactly $26 - k$ letters. What is the recursion that it satisfies?