Putnam Club Problem Sheet - April 24

Warm-up

Putnam 2008 A2. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Putnam 2006 A2. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

Putnam 2005 A2. Let $\mathbf{S} = \{(a, b) | a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

(i) $p_i \in \mathbf{S}$,
(ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$,
(iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

(An example of such a rook tour for $n = 5$ was depicted in the original.)

Putnam 2020 B2. Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

Harder

Putnam 2009 B2. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

Putnam 2002 B2. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

Putnam 2012 B3. A round-robin tournament of $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

Putnam 2011 B4. In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

Hardest

Putnam 2023 A6. Alice and Bob play a game in which they take turns choosing integers from 1 to n. Before any integers are chosen, Bob selects a goal of "odd" or "even". On the first turn, Alice chooses one of the n integers. On the second turn, Bob chooses one of the remaining integers. They continue alternately choosing one of the integers that has not yet been chosen, until the nth turn, which is forced and ends the game. Bob wins if the parity of $\{k\colon \text{the number } k \text{ was chosen on the } k\text{th turn}\}$ matches his goal. For which values of n does Bob have a winning strategy?

Putnam 2022 A5. Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

Putnam 2014 B5. In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(1) A player cannot choose an element that has been chosen by either player on any previous turn.
(2) A player can only choose an element that commutes with all previously chosen elements.
(3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

Putnam 2013 B6. Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?