## SEQUENCES AND RECURRENCES (03/13/24)

**1.** (1994-A1) Suppose that a sequence

$$a_1, a_2, a_3, \ldots$$

satisfies

$$0 < a_n \le a_{2n} + a_{2n+1}$$

for all $n \ge 1$. Prove that the series

$$\sum_{n=1}^{\infty} a_n$$

diverges.

**2.** (1992-A1) Prove that

$$f(n) = 1 - n$$

is the only integer-valued function defined on the integers that satisfies the following conditions:

(1) $f(f(n)) = n$ for all integers $n$;
(2) $f(f(n+2)+2) = n$ for all integers $n$;
(3) $f(0) = 1$.

**3.** (Bratislava Correspondence Seminar, 1999) Let $F_n$ be the Fibonacci numbers, so that $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$. Suppose $P(x)$ is a polynomial of degree 998 such that $P(n) = F_n$ for $n = 1000, \ldots, 1998$. Show that $P(1999) = F(1999) - 1$.

**4.** (Leningrad Math Olympiad 1989, Grade 10) A sequence of real numbers $a_1, a_2, \ldots$ has the property that

$$|a_m + a_n - a_{m+n}| \le \frac{1}{m+n}$$

for all $m$ and $n$. Prove that the sequence is an arithmetic progression.

**5.** (1998-A4) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number $A_n$ is defined by concatenating the decimal expansions of $A_{n-1}$ and $A_{n-2}$ from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all $n$ such that 11 divides $A_n$.

**6.** (2001-B3) For any positive integer $n$ let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

**7.** (2002-A5) Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \ge 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \ge 0 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots \right\}.$$

**8.** (1993-A6) The infinite sequence of 2's and 3's

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number $r$ such that, for any $n$, the $n$th term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer $m$.

(*Side comment:* what happens for other numbers in place of 2 and 3?)