

# Putnam Club Problem Sheet - February 21

## Warm-up

**Putnam 2005 B1.** Find a nonzero polynomial  $P(x, y)$  such that  $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$  for all real numbers  $a$ . (Note:  $\lfloor \nu \rfloor$  is the greatest integer less than or equal to  $\nu$ .)

**Putnam 2007 B1.** Let  $f$  be a nonconstant polynomial with positive integer coefficients. Prove that if  $n$  is a positive integer, then  $f(n)$  divides  $f(f(n) + 1)$  if and only if  $n = 1$ .

**Putnam 2018 B2.** Let  $n$  be a positive integer, and let  $f_n(z) = n + (n-1)z + (n-2)z^2 + \dots + z^{n-1}$ . Prove that  $f_n$  has no roots in the closed unit disk  $\{z \in \mathbb{C} : |z| \leq 1\}$ .

## Harder

**Putnam 2005 A3.** Let  $p(z)$  be a polynomial of degree  $n$  all of whose zeros have absolute value 1 in the complex plane. Put  $g(z) = p(z)/z^{n/2}$ . Show that all zeros of  $g'(z) = 0$  have absolute value 1.

**Putnam 2007 B4.** Let  $n$  be a positive integer. Find the number of pairs  $P, Q$  of polynomials with real coefficients such that

$$(P(X))^2 + (Q(X))^2 = X^{2n} + 1$$

and  $\deg P > \deg Q$ .

**Putnam 2010 B4.** Find all pairs of polynomials  $p(x)$  and  $q(x)$  with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

## Devilish

**Putnam 2007 B5.** Let  $k$  be a positive integer. Prove that there exist polynomials  $P_0(n), P_1(n), \dots, P_{k-1}(n)$  (which may depend on  $k$ ) such that for any integer  $n$ ,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \dots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

( $\lfloor a \rfloor$  means the largest integer  $\leq a$ .)

**Putnam 2005 B5.** Let  $P(x_1, \dots, x_n)$  denote a polynomial with real coefficients in the variables  $x_1, \dots, x_n$ , and suppose that

$$\left( \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \dots, x_n) = 0 \quad (\text{identically})$$

and that

$$x_1^2 + \dots + x_n^2 \text{ divides } P(x_1, \dots, x_n).$$

Show that  $P = 0$  identically.

**Putnam 2008 A5.** Let  $n \geq 3$  be an integer. Let  $f(x)$  and  $g(x)$  be polynomials with real coefficients such that the points  $(f(1), g(1)), (f(2), g(2)), \dots, (f(n), g(n))$  in  $\mathbb{R}^2$  are the vertices of a regular  $n$ -gon in counterclockwise order. Prove that at least one of  $f(x)$  and  $g(x)$  has degree greater than or equal to  $n - 1$ .